KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

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The following papers were read:

Anatomy. — "*On the development of the cerebellum in man.*" (First part). By Prof. L. Bolk.

(Communicated in the meeting of March 25, 1905.)

On account of the fact that the lobulisation of the adult cerebellum of Primates generally and of man in particular, deviates in various respects from that of the remaining mammals, so that a homologisation of the lobes of the cerebellum of Primates occasionally presents difficulties, I undertook an investigation concerning the development of the grooves and lobes of the cerebellum in man, in order to try to elucidate certain obscure points in the anatomy of the Primate-cerebellum in this way. This investigation comprises some forty cerebella of human embryos, varying in length from crown to sole from 5 to 30 cm.

All the objects had been hardened in formol *in situ*; alcoholic material cannot be used for the study of this developmental process.

A complete systematic investigation of the formation of the lobes in the cerebellum of man has until now not been carried out. RETZIUS gives in his well-known standard work — Das Menschenhirn — a great number of pictures of developmental stages, also of the cerebellum, but a reasoned explanation to them is lacking. Of an earlier period we moreover mention the communications of KÖLLIKER and SCHWALBE, of more recent date those of KUITHAN, ELLIOT SMITH and CHARNOCK BRADLEY. In general, however, these investigations have been made with material which for this purpose was insufficient and as a consequence of this, opinions have become current which I have found to be wrong. This is more especially the case with the view concerning the way in which the sulcus horizontalis develops. Particularly with a view to the individual variations which arise especially in the later period of the lobulisation, it is essential to carry out the investigation with an extensive material, if we want to form a clear and continuous idea of the process and if we desire to distinguish well what is norm here and what exception.

In the morphogenetic process of the human cerebellum three periods may be distinguished, the first period is that of the development of the "cerebellar lamella" until the appearance of the first cortical groove; the second and third periods are those of the formation of grooves and lobes, during which, in the second period, those grooves appear which are generally characteristic for the cerebellum of mammals, in the third, the specific grooves and lobes of the Primate cerebellum. In this first communication only the first and second periods will be described.

Fig. 1, 2, 3 and 4 are sufficient to give an idea of the development of the "cerebellar lamella" until the time of appearance of the first groove. Fig. 1 has been taken from a foetus of 5 cm. length from crown to sole. The curvatures of the pons and neck have reached their maximum. The cerebellum appears as the already fairly thickened "cerebellar lamella" of MIHALCOVICS. It is remarkable that the thickening is turned intraventricularly in man whereas in the rabbit and pig (CHARNOCK BRADLEY) and the sheep (KUITHAN) it is exactly the extraventricular face which is most prominent.

From figs. 2*a* and 3*a* it appears that the convexity of the intraventricular plane becomes greater and greater, while the outer plane is only slightly vaulted. As a consequence of this, the cerebellum has in Fig. 3 acquired a triangular shape in the section with one extra- and two intraventricular planes; of these latter one faces

basalwards, the other caudalwards. The original caudal edge of the "Lamina cerebellaris" has consequently been shoved upward together with the insertion of the epithelial ventricular roof. Since in the mean time the plica-chorioidea has been formed, the peculiar condition has arisen, described by His for the human cerebellum, of which Kuithan, however, denies the existence, since he could discover no trace of it in the sheep, although a complete developmental series was at his disposal. The anterior plait of the plica chorioidea, the so-called lamina chorioidea, is now stretched parallel to the intraventricular plane of the cerebellum which is turned backwards and this gives origin to a slit-shaped space between this plane and the lamella mentioned. His described this in this way that the lamina chorioidea partially encloses the cerebellum like a sac and how far this is the case appears from fig. 3*b* where the cerebellum is seen laterally. Now in fig. 4 nothing is found any longer of this condition, the plica chorioidea is now inserted at the edge of the cerebellum which now is turned to the back. Also in this respect I can confirm the observations of His against those of Kuithan, that namely the lamina chorioidea lays itself upon the intraventricular plane and coalesces with this. Through this the latter has become an extraventricular plane and the plica chorioidea has obtained a new secondary line of insertion with the cerebellum. At the same time the outer plane has in this way become convex, the inner plane shows as a first indication of the "tent" in its posterior part a shallow groove which is to be distinguished as Incisura fastigii. The primitive line of insertion of the lamina chorioidea has to be sought in fig. 4 at the top of the extraventricular plane, laterally it lies more caudally, as follows from fig. 3*b*. This developmental stage of the cerebellum in man seems, by the peculiar way in which it thickens, to differ fundamentally from that of other mammals, where the cerebellar lamella retains in the section a more flattened lenticular shape.

At first the thickened lamina cerebellaris has the shape of a semiring, standing vertically on the anterior part of the longitudinal axis of the rhombencephalon and laterally passing into the still smooth regio pontis without a sharp border (fig. 2*b*). Soon the lateral parts of the lamina cerebellaris show a fairly strong clublike swelling (fig. 3*b*) by which a clear demarcation between cerebral base and cerebellum is formed. These lateral swellings remind us of the bilateral origin of the cerebellum in lower vertebrates (observed e.g. by Schaper in Teleosteans). Yet this lateral demarcation is only temporary; as soon as the pons begins to differentiate, it disappears again and arises

anew only at a much later stage when the cortex has already become amply lamellised.

In the mean time, during the thickening of the lamina cerebellaris, developmental phenomena have taken place in the bordering region between Mesencephalon and Rhombencephalon, giving rise to the formation of Plica encephali dorsalis (KUPFER), Isthmus rhombencephali and Velum medullare anterius. In the youngest stage represented (fig. 1) the anterior edge of the Lamina cerebellaris passes directly into the mesencephalic roof, only the posterior edge of this latter is a little inwardly invaginated. An Isthmus rhombencephali or Plica encephali dorsalis do not yet exist. In the next stage of development (fig. 2*a*) the mesencephalon has obtained a clearly defined posterior wall, vertical to the roof; the inward invagination of the posterior edge of the roof is still in existence and is partly visible in fig. 3, but has disappeared in fig. 4 on account of the thickening of the posterior wall of the mesencephalon. In fig. 2 the plica encephali dorsalis has developed, bordered in front by the posterior wall of the mesencephalon, at the back by the lamina cerebellaris. The formation of the Plica is accompanied by a rotation of the Lamina cerebellaris, the anterior edge of which is now no longer situated at the front but below and as a consequence of this the Isthmus rhombencephali is now also indicated in principle. Next the bottom of the plica encephali dorsalis becomes broader, there arises between the thickened lamina cerebellaris and the mesencephalon a thin middle plate (figs. 3*a* and 4), the first origin of the velum medullare anterius. The further details of this stage and the following stages will be extensively described elsewhere.

The lobulisation of the cerebellum in the second stage is characterised by the fact that the grooves which divide the surface of the cerebellum into several regions originate with a single exception in the median plane and from there extend laterally. These interlobular grooves are consequently unpaired with one exception and divide the foetal cerebellum of man into a number of lobes, which can be homologised without difficulty with those which I learnt to be typical for the adult mammalian cerebellum.

The median section of a cerebellum with indications of the grooves that appear first, is given in fig. 5. The incisura fastigii has been shifted more to the front compared with fig. 4. On the extraventricular plane two grooves can be clearly distinguished, one, a little rostrally from the top of the cerebellum, another at a short distance from the margo myelencephalicus. Which of these two arises first I have not been able to make out, evidently they both arise about simultaneously,

since in three cerebella of this stage I found both of them already present in each (total length of the foetus 8 to 10 cm.). The anterior groove is the sulcus primarius (1), the typical principal groove, easily recognised in every mammalian cerebellum, separating the two lobes of the cerebellum, the lobus anterior and lobus posterior. The posterior groove is the sulcus uvulo-nodularis (*mihi*) (sulcus postnodularis of ELLIOTT SMITH, sulcus praeuvularis of ZIEHEN, Fissure IV of CHARNOCK BRADLEY). It borders the nodulus in front, i. e. in the direction of the mesencephalon. Between these grooves a still shallow depression is visible on the upper part of the posterior plane, the first indication of the fissura secunda (ELLIOTT SMITH, *mihi*, sulcus inferior anterior of ZIEHEN, fissure *d* of CHARNOCK BRADLEY). The cerebellum, seen at this stage from behind, is somewhat biscuitshaped (fig. 6) and lies with the front planes of its lateral parts against the occipital poles of the hemispherical vesicles. Besides the two mentioned grooves, proceeding from the median line, the cerebellum possesses at this stage already a sulcus which is bilaterally symmetrical and lies at a short distance of the margo myelencephalicus. This groove (*p*), which develops in a latero-medial direction is the homologon of the groove which I have distinguished in the mammalian cerebellum as fissura paraflocccularis. It borders in front the already slightly prominent so-called recessus lateralis. The anterior wall of this recessus lateralis has been distinguished by KÖLLIKER as gyrus chorioideus. It seems to me that the name "Gyrus floccularis" is more characteristic, since from this narrow cerebellar seam which is already marked out at so early a stage, the flocculi are later formed.

In a successive stage (Figs. 7 and 8) the sulcus primarius (1) has become deeper and the fissura secunda (2) has become a distinct groove; moreover a first secondary groove has arisen in the lobus anterior. Later this lobus is separated into four small lobes by three grooves; I have not been able, however, to make out which of these three is the first to appear. That the sulcus in the lobus anterior, seen in figs. 6 and 7, which is first in appearing, is really the groove, distinguished by ELLIOT SMITH as sulcus praeculminatus, I have not been able to confirm, while also from a comparison of my human cerebella of this stage with corresponding figures given by CHARNOCK BRADLEY for the rabbit and pig, it appears that this first groove lies in the lobus anterior of man farther away from the sulcus primarius than with the two animals mentioned. So I cannot decide whether this first groove in the lobus anterior in man is homologous with the "Fissure 1" of CHARNOCK BRADLEY. Also in the lobus posterior a new groove has appeared in the median line, between

the sulcus primarius (1) and the fissura secunda (2). This groove, indicated in the following figures by 4, is the sulcus praepyramidalis (mihi) (sulcus inferior posterior of ZIEHEN, fissure suprapyramidalis of ELLIOT SMITH, fissure III of CHARNOCK BRADLEY). This fissura praepyramidalis borders in front the pyramis and soon reaches the length of the fissura secunda. This latter can in its further development lengthen itself regularly in a lateral direction, or otherwise there independently arises (fig. 8*b* 2') in the hemisphere at a short distance above the fissura paraflocculari (*p*) a groove which soon becomes confluent with the fissura secunda. While at the same time the recessus laterales bend out further, the fissura paraflocculari (fig. 7*b*, 8*b*, *p*) becomes confluent with the sulcus uvulo-nodularis, by which the gyri flocculares form with the nodulus a part which is marked off from the remainig cerebellum. ELLIOTT SMITH mentions that the fissura paraflocculari can also flow together with the fissura secunda (2). This observation I can confirm for other mammals on account of the structure of the adult cerebellum; with the embryonic material of man I have not observed such a case. At a later stage the fissura secunda does terminate, above in the fissura paraflocculari.

In the cerebellum of a foetus of 13 cm. the hemispheres are no longer swollen balloonlike, but have, when seen from behind, obtained the more angular form which now is characteristic for them during a longer period of development (fig. 9*a*). The median zone is still a little depressed, even in the posterior part of the lobus anterior. The sulcus primarius (1) lies still relatively far at the back, the sulcus praepyramidalis (4) has already pretty far advanced into the hemispheres, but in such a way that the lateral parts with the transversally proceeding middle part form an obtuse angle, with the opening downwards. This peculiar shape forms during the successive stages of development, in which the interpretation of the grooves is not always easy, an excellent diagnostic for the sulcus praepyramidalis. The fissura secunda (2) has advanced as far as the lateral wall of the cerebellum so that the regio tonsillaris (fig. 9*a t*) is now bordered on all sides. This region is always more or less swollen in the shape of an egg. The gyrus floccularis is divided by a longitudinal groove into two small lobes. At this stage consequently the uvula with the appertaining lateral parts and so also the nodulus with its lateral regions are already differentiated in principle. This rapid developmental process contrasts strongly with the still very simple condition found in the remaining part of the lobus posterior and supports to some extent the opinion of ELLIOTT SMITH who looks upon the complex of uvula with tonsils, nodulus with flocculi, as a more independent

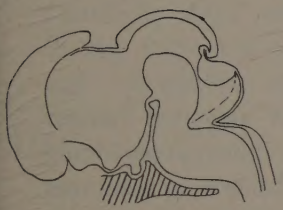


Fig. 1.

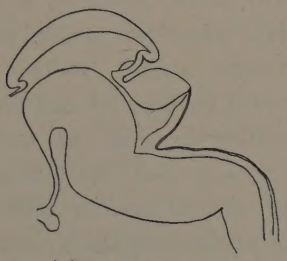


Fig. 2a.

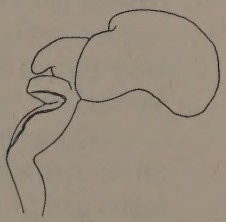


Fig. 2b

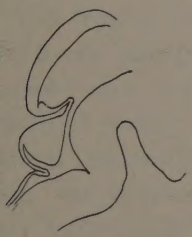


Fig. 3a.

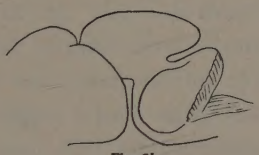


Fig. 3b.

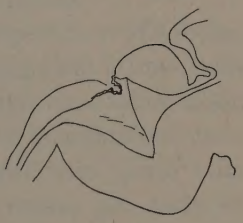


Fig. 4.

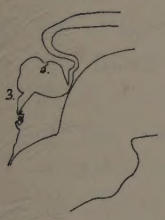


Fig. 5.

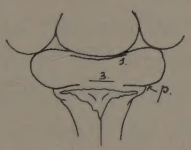


Fig. 6.

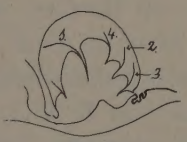


Fig. 7a.

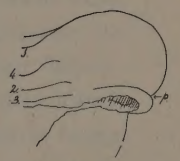


Fig. 7b

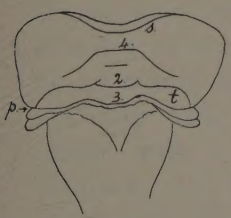


Fig. 9a.

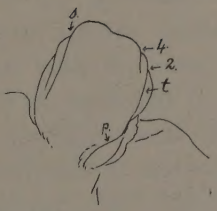


Fig. 9b.

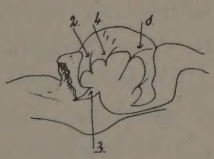


Fig. 8a.

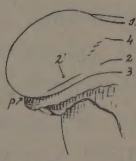


Fig. 8b.

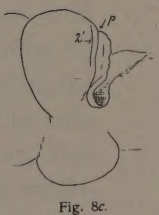


Fig. 8c.

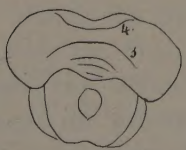


Fig. 10a

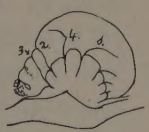


Fig. 10b.

lobe of the cerebellum. Then the peculiar surface division in the median line between sulcus primarius (1) and fissura secunda (2) deserves notice. The sulcus praepyramidalis (4), namely, at first always divides this region into two unequal parts; the lower half, the greater, situated between sulcus praepyramidalis (4) and fissura secunda is the origin of the pyramis, while from the very narrow upper half, situated between sulcus primarius (1) and sulcus praepyramidalis, must originate: declive, folium vermis and tuber vermis. In this respect a parallelism can be noticed between the phylogenetic and ontogenetic development of the cerebellum. For the narrow region between sulcus primarius and sulcus praepyramidalis is homologous with that lobulus which in the median section of the mammalian cerebellum I have distinguished as lobulus C_2 and which only in the Primates attains a very strong development.

The frontal plane and median section of the cerebellum of a foetus of 15 cm. are given in fig. 10. This stage of development is important because now the foetal human cerebellum shows the same lobulisation which I learnt to be the fundamental type of the mammalian cerebellum generally, a stage which only lasts a short time, since now soon the grooves appear that characterise the Primates generally or the Anthropoids and man more particularly and the homologa of which are missed with other mammals. For as will be seen from fig. 10*b*, now in the median section, as well the lobus anterior as the lobus posterior, is divided by three grooves into four lobuli. With the mammalian cerebellum I have distinguished the four lobuli of the lobulus anterior as lobulus 1, 2, 3 and 4, the latter being situated immediately before the sulcus primarius, the four lobuli of the lobus posterior I distinguished as lobulus *a* (homologous with the nodulus), lobulus *b* (homologous with the uvula), lobulus C_1 (homologous with the pyramis) and lobulus C_2 (homologous with the complex of declive, folium vermis and tuber vermis). It will be seen by a comparison with the investigations of CHARNOCK BRADLEY, that the stage with man, sketched in figs. 9 and 10, has a strong resemblance with a developmental stage which the cerebellum of other mammals (pig and rabbit) traverse before the final lobulisation of the cerebellum. However much the Primate cerebellum may in its final form differ from that of other mammals the groundplan of its lobulisation is, as is evident from the now sketched period of development, not different from that of other mammals. In the next following stage, however, it follows a development of its own, grooves occur, introductory to the lamellisation of the cortex, which are specific for the Primates and which will be described in the second communication.

Botany. — *"Some observations on the longitudinal growth of stems and flower-stalks"*. By Prof. E. VERSCHAFFELT. (Communicated by Prof. HUGO DE VRIES).

(Communicated in the meeting of March 25, 1905).

Superficial observation already shows that in many cases the growth of stems, leaf- and flower-stalks is greatly dependent on the organs which they bear: buds, leaf-lamina, flowers. When these latter are removed the growth of the axial parts is generally arrested and they even die after a shorter or longer time.

In literature I have not found any investigations mentioned, attempting to analyse this phenomenon more closely; e.g. in the case of flower-stalks, to find out whether excision of certain parts of the flower had as much influence on the growth of the stalk as the removal of the entire flower. I have now been able to make this out for some vernal plants by measurements of growth and I shall in what follows give a short account of the results.

I chose preferably flowers for this purpose, since here at the top of the same spindle organs of different physiological functions occur together and so the experiments admitted of greater variety. In one case, that of *Eranthis hiemalis* Salisb., I shall describe the course of the investigation and its results a little more in extenso; the other examples will be more briefly dealt with.

The stem of *Eranthis*, as will be known, bears at its top a single flower and close under it, united to a sort of broad collar, a whorl of three green sitting parted leaves. As long as the stem is still under the ground, its top is sharply bent downward and the still perfectly closed flower hangs down, protected by the three leaves, still yellow then, which envelop it. As soon as the top of the stem has come above the ground and also the flower has come free, this latter raises and soon unfolds itself; then the basal collar spreads out and turns green. The measurements of growth were made in the stage between the period when the stem is not yet visible above the ground and that, in which, after the petals and stamens have fallen off, only the fertilised pistils remain. About this time the longitudinal growth stops. Whether afterwards, during the ripening of the fruits, a new period of growth begins, as in other plants, I have not investigated.

The plants, serving for the investigation, were placed in a hothouse of the Botanical Garden at Amsterdam, in which the mean temperature was 20° C. and in which the specimens developed very rapidly and entirely normally.

I will first show by a few examples that the presence of the organs on the top is necessary in order to cause the stem to grow normally in length.

The stem of an *Eranthis* was on February 4, 1905, 40 mm. long, measured from the base near the rhizoma to the junction of the leaf-whorl. Placed in the hothouse the plant was at first measured daily, afterwards every other day; for briefness' sake I shall here only give the length reached by the stem after every four or five days.

Date	4.2.05	8.2	13.2	17.2	22.2	26.2
Length in mm.	40	89	135	154	162	162

In the same time the development of a stem on which leaves and flower had been cut away, was:

Date	4.2	8.2	13.2	17.2
Length	49	52	54	55

Another example of growth with a normal stem:

Date	5.2	9.2	13.2	17.2	22.2	26.2
Length	44	98	128	145	150	150

and of a stem, bereft of leaves and flowers:

Date	5.2	9.2	13.2	17.2
Length	97	103	104	104

Whereas with normal *Eranthis*-stems the top with the flower on it, had in the hothouse after a couple of days, entirely erected itself, on the other hand the hook-shaped curvature of the stem without flower or leaf-whorl, partially remained and it was only very slowly that its extremity raised itself to some extent. This need cause no wonder, if it is remembered that the disappearance of this curvature is caused by asymmetrical growth of the top of the stem.

Now in a series of *Eranthis* plants the organs on the top of the stem were only partly removed; e.g. the three green leaves, the petals, the stamens, the pistils. The length of the stems was measured from day to day. The result of these experiments has been very clear. As long as the green leaves remained undamaged, the growth of the stem might be called normal. At the utmost the stem remained a little below its normal length if the whole flower or certain parts

of it were cut away. On the other hand the growth was very considerably impeded by removing the whorl of green leaves. This will be seen from the following measurements.

Eranthis-stem, on which only the three leaves under the flower have been preserved, the flower itself having been removed:

Date	7.2	11.2	15.2	19.2	22.2	26.2
Length in mm.	51	107	134	141	141	141

Another example of the same case:

Date	6.2	10.2	13.2	17.2	22.2	26.2
Length in mm.	58	104	129	135	135	135

Eranthis-stem of which the basal whorl has been cut away, the flower remaining intact:

Date	6.2	10.2	13.2	17.2	19.2
Length in mm.	86	96	97	100	100

Another example of the same case:

Date	7.2	11.2	15.2	17.2
Length in mm.	59	72	74	74

Hence a stem which had been bereft of its flower grew in length in a period of twelve days 176 % in the first and 133 % in the second experiment, this increase in length being only 16 % and 25 % respectively in the same time with a stem on which the flower had been preserved but the whorl of leaves removed.

The influence which the presence of the leaf-whorl has on growth follows clearly enough from this. Also in the other cases which I investigated, the growth of stems that bore flowers only, may have been a little greater than of stems from which the leaf-whorl as well as the flower had been removed, it is certain, however, that the longitudinal growth is chiefly regulated by the presence of the green leaves.

A related fact is that after removal of the leaf-whorl the flower raises itself only very slowly and often only partly.

Although the supposition is not very probable, it might be presumed that the observed effect of the three leaves is caused by the circumstance that they have to provide the stem with food. That this

is not the case follows from the fact that the same results are obtained in the dark and that consequently the presence also of the non-assimilating leaves renders a strong longitudinal growth of the stem possible, which does not occur if only the flower is preserved on the top. It will be superfluous to mention figures in this respect.

No more does it appear necessary to give in extenso the measurements proving that removal of the pistils, the stamens or the petals has with *Eranthis* little or no influence on the longitudinal growth of the stem. On the other hand it is not superfluous to remark that the leaf-whorl must be pretty completely cut away if we want soon to arrest growth. The three green leaves namely show basal growth themselves and if their foot is not damaged, this latter may appreciably grow in size in the course of a few days; at the same time the stem continues growing in length.

Example: foot of the three green leaves kept; also the flower intact.

Date	8.2	11.2	15.2	20.2	26.2
Length in mm.	54	81	113	145	145

Already on the 13th the leaf-whorl had considerably grown out; at the edge nothing of the nature of a wound could be seen any more. In the same time a stem of 102 m.m. length on which the leaf-whorl had been completely cut away, the flower remaining intact, had only reached a length of 117 m.m.

If one should be inclined to think that not the presence of the whorl of green leaves but the intact condition of the junction of the leaves on the stalk is the principal point here, I must remark that of this junction zone a layer of tissue may be removed all round without the longitudinal growth being materially affected. Also from the somewhat vaulted receptacle a part may be removed or the middle part may be hollowed without any other consequences than would ensue on the plucking off of the floral parts situated on it.

Finally we remark that *Eranthis*-stems, cut off near the junction on the rhizoma can continue growing for days when they are put with their feet in water and then show the same behaviour as whole plantlets. Besides, the presence of one out of the three green leaves is sufficient to render a considerable growth in length of the stem possible; e.g. lengthening from 53 to 89 mm. in two weeks. That also with *Eranthis*-leaves the growth of the leaf-stalk depends on the presence of the leaf-disk will now be obvious; I have ascertained myself of it by measurements, however.

Galanthus nivalis L. enables us to observe phenomena of a different kind in this same respect. With this plant also, the stem terminates in a single flower which, however, when it is fully developed and unfolded, hangs on a thin, limp, flower-stalk. This is implanted on the top of the stem, where also two coalescent bracts are found which enveloped the flower-bud before its unfolding. Hence we must here investigate the influence of the terminal organs on the growth of the stem as well as on that of the flower-stalk.

Concerning the longitudinal growth of the stem, we find that it is completely independent of the presence of the flower. A single example will suffice to show this. The stem was measured from the point where it appears from the bulb to the implantation of the bracts; these latter still surrounded the flower-bud; in *a* the plant remained undamaged; in *b* bracts and flower were cut away to the foot.

	Date	13.2	16.2	20.2	23.2	26.2
Length in mm.	*a.*	90	133	157	161	162
	b.	46	60	90	105	108

On the other hand, the growth of the flower-stalk stops as soon as the flower is removed. The influence of the flower on this organ is even so great that already after a couple of days the stalk of cut flowers turns yellow at the top and soon dies from above downward. The measurements show that the ovary plays if not a preponderant, yet a considerable part here. So the flower-stalks of flowers which already opened, grew from 28.2.05 to 6.3.05, in two cases from 16 and 14 mm. to 23 and 24 mm.; a flower of which the perianth was removed, in the same time from 17 to 21 mm., while two flower-stalks without their flowers measuring 20 and 14 mm. had reached 22 and 16 mm. the next day, but after that died off. Cutting the stamens has no great influence on growth; yet growth remains very small if stamens as well as perianth are removed, so that with *Galanthus* the ovary regulates the growth of the flower-stalk to a great extent but not exclusively. On the other hand the flower-stalk remains alive as long as the ovary is still present on its top.

Exactly the same behaviour is shown by *Narcissus Pseudo-Narcissus* L., where the stem continues growing when the flower is cut, but the flower-stalk stops growing and dies, if the ovary is wanting. I may add here that for the growth of the stem it makes no difference whether its top is cut above or below the

swelling occurring at the point where the bracts and flower-stalk are implanted, so that this zone also has no importance for the growth of the parts under it. Also stems of *Galanthus* and *Narcissus*, cut in the basal part and hence separated from the bulb, or even parts of them, if they were taken from plants with their flower-buds still closed, continue to grow vigorously whether the flower-bud be present or not.

Tulipa Gesneriana L. shows something different again. Here the flower is born by a leafed stem; the internodes which are placed near the base stop growing sensibly towards the time that the flower becomes visible from the outside and is about to open. At this stage, however, the upper internode with the flower at the top, still grows considerably in length. For this the presence of the flower is absolutely necessary. The upper portion of the stem is arrested in growth and gradually dies off as soon as the flower is cut off.

Example: *a*. flower present; *b*. flower removed. Only the upper internode measured.

	Data	6.3	8.3	13.3
Length in mm.	*a*	42	53	83
	b.	41	42	44

From the following measurements the significance of the various floral parts may be seen:

a. perianth removed.
b. stamens removed.
c. pistil removed.

	Date	6.3	8.3	13.3
Length in mm.	*a*.	36	41	45
	b.	46	63	70
	c.	41	51	68

Although removal of each of the individual whorls of organs, partly suppresses the growth of the upper internode, yet it is seen that the petals have the greatest influence here. The above is only an example chosen from several concordant measurements.

Finally some observations were made with *Crocus vernus* All. Since the ovary lies fairly deep here, hidden in the tube formed by

the green leaves and the bracts round them, plants that had been cut open had to be used for the measurements, in which the flower was laid bare over its full length. For this purpose flowers were chosen which were still surrounded by bracts and entirely closed and the top of which became just visible above the ground. It appeared, however, that at this stage the stem on which the flower is situated, had reached about its full length and only grew a few millimetres more. The further longitudinal growth which is very considerable and brings the flower above the ground is nearly wholly caused by the corolline tube between the ovary and the loose slips of the perianth. Only to this stage I paid attention. Some measurements of the corolline tube may follow:

a. flower undamaged.
b. corolline lobes removed.
c. corolline lobes, stamens and pistil cut away at the upper end of the coalescent corolline tube.

	Datum	8.3	9.3	11.3
Length in mm.	*a.*	46	101	108
	b.	55	84	84
	c.	31	72	72

So removal of the terminal organs has not remained without influence on the growth of the corolline tube, but has not been able to check it to the same extent as in the preceding cases.

It deserves notice that removing the anthers and stigmas did not prevent the stamens and styles to reach about their normal length.

Summary. The investigation has shown that the normal longitudinal growth of the stem with *Eranthis hiemalis* is only possible when the whorl of leaves at the top is present, while the flower exercises no influence on it. This latter is also the case with the stem of *Galanthus nivalis* and *Narcissus Pseudo-Narcissus*; the flower-stalk however, in these two plants, is checked in growth as soon as the flower is cut, the ovary proving to be of especial importance. With *Tulipa Gesneriana* it is chiefly the perianth that rules the longitudinal growth of the upper internode; with *Crocus vernus*, finally, the growth of corolline tube, stamens and style is in a high degree independent of the presence of petal lobes as well as of anthers and stigmas.

Zoology. — "*On the Structure of some Siliceous Spicules of Sponges.* I. The styli of *Tethya lyncurium*, by Dr. G. C. J. VOSMAER and Dr. H. P. WIJSMAN, Professors at the Leiden University.

(Communicated in the meeting of April 22, 1905).

After SCHWEIGGER (1819) had demonstrated that the spicules of sponges in some cases do not consist of calcium carbonate, GRANT (1826) found them to contain silica, and BOWERBANK (1841 *a*) showed that, in addition to the silica some organic matter is present. He reached this conclusion through the fact that the spicula when heated, were partly carbonised. KÖLLIKER (1864) remarked that the brown or black colour, produced by heating, is certainly not only due to carbonised organic matter; examined in reflected light the heated spicula appear white, and the dark spots seen in transmitted light are, therefore, partly due to inclosed air. THOULET (1884) found no organic matter and concluded: "les spicules sont donc constitués par de la silice pure", which he compares with opal. SOLLAS (1885) likewise finds that the silica resembles opal. It is now generally accepted that the spicules of siliceous sponges consist of some kind of opal; but that in some way or other, organic matter is also present. So far here is a general agreement of opinion but the chemical analyses which have been carried out show considerable differences as to the quantity of water, combined with the silica as a gel. The formulae, given for the composition vary from $2\ (SiO_2) + H_2O$ to $5\ (SiO_2) + H_2O$, but must be considered as mere failures. F. E. SCHULZE (1904) comes to the result: "dass, entweder die Siphone keinen bestimmten konstanten Wassergehalt haben, oder dass die organischen Zwischenlamellen .... einen je nach der vorgängigen stärkeren oder geringeren Austrocknung wechselnden Gehalt an Wasser haben".

It is certain that even in the best cases, the quantity of organic matter is so little that it cannot well be ascertained. Its presence can, however, be proved by treating the spicules with hydrofluoric acid. But there is also some disagreement as to the nature of this matter and the exact place in the spicules where it is met with. We shall see that different kinds of spicules vary in this point.

In addition to the silica, which behaves like some kind of opal, and which we propose to call *spicopal*, and the organic matter which F. E. SCHULZE called *spiculine*, — modifying the original term used by HAECKEL — in some spicules there have been found traces of Na, K, Cl, Fe, Mg and Ca, but in such slight quantities that they can be left out of consideration for the moment.

As to the structure of the spicules, GRAY (1835) had found them in *Hyalonema* to consist of layers, which became conspicuous by heating. These layers concentrically surround a "central canal", which is filled out, as KÖLLIKER (1864) has shown, by an organic mass, the axial rod. CLAUS (1868) found that the silica which directly surrounds this central rod, is homogeneous; he called this homogeneous cylinder the axial cylinder. According to MAX SCHULTZE (1860) the longitudinal striae, which become conspicuous especially after heating, are due to the fact that layers of silica alternate with very thin layers of organic matter; the first are, after SCHULTZE, isotropic, the second anisotropic. The outer layer is generally found to be of organic nature.

In all these cases, the investigators described some special kind of spicule; naturally they have chosen very large spicules. GRAY, CLAUS and MAX SCHULTZE studied the large rods of Hexactinellida, such as *Hyalonema* and *Euplectella*. It may be asked, how far their results hold good for spicules of other sponges.

KÖLLIKER had already found that not in every case the axial thread is conspicuous. Also it has not been possible to demonstrate in every case alternating layers of spicopal and of organic matter, not even where longitudinal lines are evident. There is a great confusion with regard to the presence of a so-called spicular sheath. Any accurate determination of the refractive power of different spicules is likewise wanting.

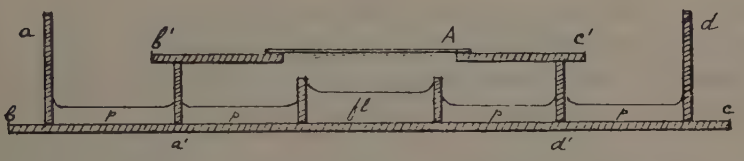
It seems, therefore, desirable to get some more information about these subjects. Since F. E. SCHULZE (1904) studied the enormous spicula of *Monorhaphis*, it appears useful to investigate, whether spicules from other groups agree with them as to their structure, and their chemical and physical properties.

We began our examination by the large styli of *Tethya lyncurium*. After bringing this point to some certainty, we have compared the results with those obtained from other species.

In the first place we tried to answer the question: Do the styli of *Tethya* contain other organic elements than the central rod, either as a sheath, or as layers between the spicopal, or as both of them.

Of course, the method which at first presents itself for the detection of organic matter, is the dissolution of the silica by means of hydrofluoric acid. Former investigators who applied this reagent, have omitted to give an accurate description of their experiments. SOLLAS (1888 p. XLIX) put the isolated spicules into a drop of water and added a drop of hydrofluoric acid. On doing this, one generally sees that the silica is dissolved and that the central rod remains.

Adding less hydrofluoric acid, the process does not go to the end. As it is necessary to cover the slide by means of Canada balsam (SOLLAS) in order to preserve the object glass from the disastrous influence of the vapours of hydrofluoric acid, it is difficult to vary the acidity of the fluid, in which the spicules are mounted. Also it is impossible in this way, to exclude the influence of the glass. We had therefore, to construct an apparatus allowing the concentration of the hydrofluoric acid to be varied without danger to the lenses of the microscope. At first we tried ebonite, in combination with glass, covered by a layer of celluloid, which is sold in solution under the name of zapon. But as this method did not satisfy our purposes, we tried another and we think that we have found a good and rather simple device. The adjoined figure needs little explanation. Out of a sheet of trans-



parent celluloid, 1 mm. thick, is constructed a case, *abcd* [1]); the bottom measures 6×10 cm., the height is 1.5 cm. In the midst of the case a circular rim of celluloid, high 5 mm., is joined to the bottom. Another case of celluloid *a'b'c'd'* measures 4.5×6 cm. bottom and 8 mm. height. In the middle of the bottom a square of 2×2.5 cm. is cut out. This opening is covered by a sheet of thin celluloid, no more than 0.5 mm. thick, measuring 2.5×3.5 cm. This thin sheet *A* is joined with *b'c'* air- and watertight by means of soft paraffin. The spicules under examination are placed on the bottom of the case *a'b'c'd'*, e. g. in a drop of water. To flatten this drop it is covered by a very thin film of celluloid. In the exterior part of the case *abcd* is poured out some liquid paraffin *p*, in the interior commercial hydrofluoric acid (*fl*), diluted with 3 or 4 parts of water. Now the case *a'b'c'd'* is reversed and put into *abcd*. As we have found that hydrofluoric acid in the gaseous state after some time

[1]) The carefully cut and cleaned sheets of celluloid are easily united by means of acetone. The parts that should be united are pressed together, and by means of a small brush a little drop of acetone is applied. The celluloid immediately sticks together. The corners are afterwards, to diminish the chance of leakage, cemented another time with zapon. Celluloid is got at the D. Celluloid Fabrik Leipzig.

[1]) Very thin films of celluloid are got by pouring out zapon on a glassplate, in the way that collodion plates are made, and tearing it off the glass after drying.

passes through thin strata of celluloid, it might be expected that the very thin layer of celluloid used in the above manner would be no obstacle for the acid to attain the spicules. Even the retaining of the reaction in this way is an advantage, as it may now be observed without the use of very dilute solutions.

In this way we have constructed a little apparatus which has answered to our purposes in several respects, viz: absence of glass, slow reaction, rather great security for the lenses of the microscope, and the possibility of interrupting the reaction at any moment.

As we found it unmaterial whether the spicules (obtained from sponges preserved in alcohol) were isolated by means of artificial gastric juice (after some days at 35° C) or by boiling with hydrochloric acid, we preferred the latter method. Such spicules, having been boiled with hydrochloric acid for some minutes, washed out repeatedly with water, either with the aid of a centrifuge or not, and dried afterwards at the ordinary temperature, are the objects investigated by us, if another treatment is not expressly mentioned. By placing some spicules in a drop of water, and covering them with the thin film of celluloid, one can first study under the microscope whether they are normal in their aspect, uninjured etc. Also, and *Tethya* is a proper object for this, which spicules are open, which closed. Then we expose the preparation to the vapours of the hydrofluoric acid. The commencement of the reaction is more or less retarded, depending upon the concentration of the acid, the quantity of water and the thickness of the film of celluloid; but at any rate the spicules begin to be dissolved after some 30 or 50 minutes.

It is best to give attention to the fractures, as the reaction is here to be seen at first.

Bütschli (1901) already remarked that the dissolution of the silica may occur in different ways. We can confirm this observation for the styli of *Tethya*. Observing what happens at the broken end of a spiculum, the opal surrounding the axial thread is seen to be hollowed out in the shape of a cone. The top of the cone is very sharp, and becomes still sharper if the reaction proceeds. Bütschli says (l. c. 258—259): "Man könnte wegen dieser so häufigen Bildung einer trichterförmigen Auflösungshöhle an den Enden auf die Vermuthung kommen, dass die Angreifbarkeit und Löslichkeit der Schichten von aussen nach innen, gegen den Achsenfaden successive zunehme. Eine solche Annahme scheint jedoch zur Erklärung der Erscheinung nicht nöthig, vielmehr dürfte sie sich schon daraus hinreichend erläutern, dass die Flusssäure almählich in den geöffneten Achsenkanal eindringt und gleichzeitig auch in dem Masse

stärker wirkt, als der Achsenkanal durch Auflösung erweitert wird, indem dann eine grössere Menge der Säure zur Verfügung steht." We believe, on the contrary, that it really follows from the observation that the inner parts of the spicopal are more easily dissolved than the outer ones. For we see the sharp conical funnel long before any trace of reaction is to be seen on the rest of the fracture. The borders remain intact for a considerable time. And we get the same view by boiling in a solution of caustic potash. Only when the conical hole has attained a certain depth, the dissolution of the spiculum from the exterior commences. Finally there remains a kind of tube, formed of silica in which the axial thread is laying isolated, until all the spicopal has entirely passed into dissolution.

If we study the way in which the hydrofluoric acid acts on completely intact spicules, some difference may be seen, according as we have to do with sharply pointed needles or with those of which the apex is rounded off. Blunt styli, i. e. transitions to strongyli, resist the acid for some time; but once the dissolution has begun from the exterior, the process proceeds regularly, the spiculum becomes thinner and thinner.

It seems that in pointed styli, the apices are first attacked; in such cases we see the axial thread gradually coming free by external dissolution of the spicopal. Sometimes it may be observed that, in addition to the dissolving process as described above, a hollowing out along the axial thread takes place. In other cases, however, this is not seen, and we get half dissolved spicules in which the axial thread is partly freed, partly enclosed in a coat of spicopal, thus strongly resembling whips; the more so as the thread is generally flexible, whereas the rest is still straight and rigid.

Bütschli has already remarked that there are sometimes seen "durch lokale stärkere Auflösung der Kieselsubstanz zellenartige Vertiefungen der Nadeloberfläche". "Indem diese Vertiefungen schliesslich zu Löchern werden, die bis zum Achsenkanal reichen, wird dieser der Flusssäure zugänglich und nun beginnt von diesen Löchern des Kanals aus . . . . die innere Auflösung der Kieselsubstanz unter Entwicklung zweier trichterförmiger Höhlen . . . ." This observation we can confirm; we consider it another proof that the spicopal in the neighbourhood of the central thread is more easily dissolved than the peripheric mass; the observation can hardly be explained in another way.

Spicules, dissolved in the described way, show, that there remained not only after the dissolution of the spicopal an organic central thread, but also a very thin coat, which covered the exterior layer of the spicopal. This coat, which represents the true spicule sheath, is

extraordinarily delicate; consequently it is easily torn or shrunk. Still, we are convinced that it exists, but the examination must be carried out with the utmost care. In some preparations we found it in the greater part of the objects. That it is not always observed may partly be due to the treatment of the spicules with hydrochloric-acid, partly by its being destroyed already during the life of the sponge. The axial thread is likewise not always visible, or at least not over the whole length of the spiculum. On carefully dissolving the spiculum it may be observed that, while the cylinder of silica gradually diminishes its diameter, a very thin line shows the dimensions it originally possessed.

When the spicules are observed in water, the limits show themselves as rather broad black bands, as the refractive index of the spicopal is considerably higher than that of water. When the dissolving process goes on, the black bands gradually approach each other, and the thin line, the optical section of the spicule sheath, becomes conspicuous. When the object is now studied in acid fuchsine, the central thread stains intensely red as it is set free, and the sheath becomes faintly reddish in the mean time. Organic layers, so called layers of spiculine, such as can easily be demonstrated in the large needles of Hexactinellida, are nowhere met with in *Tethya*. In some cases we saw something which resembled them, but in every case we could explain the phenomenon by a folding of the sheath. Consequently we conclude that layers of spiculine are absent in *Tethya*. And we cannot agree with MINCHIN (1900), who says about spicula in general (l. c. p. 40): "the mineral matter is deposited round it (viz. the axial thread) in concentric lamellae of colloid silica, alternating with lamellae of organic nature".

As to the spicule sheath, writers do not agree. What F. E. SCHULZE calls "Spiculascheide" in his last publication (1904) is not homologous with what we indicate with the name of sheath. That we notwithstanding use this term has historical reasons, as it seems to us that the word is originally used for formations belonging to the spicule itself, homologous to the product which is found in calcareous spicules, where its existence had been first demonstrated. In this sense MINCHIN applied the term, and he is the author of the newest and best general treatise on Porifera.

KÖLLIKER (1864) may be regarded as the discoverer of the spicule sheath. It seems to us beyond doubt what KÖLLIKER meant by it, although we acknowledge that his opinion is not always expressed with the utmost clearness, and that from the beginning there had existed some confusion of ideas about this organ.

KÖLLIKER says (l. c. p. 64—65), speaking about "*Nardoa spongiosa*": "Ausserdem finden sich dann noch nach der Auflösung der Spicula durch Essigsäure, zahlreiche Lücken, welche diese Bildungen enthalten, die allen von einer scharfen Linie begrenst sind, wie bei *Dunstervilia*. Bei *Nardoa* glaube ich mich davon überzeugt zu haben, dass diese scharfe Linie der optische Ausdruck einer selbständigen Scheide der Spicula ist . . ." What KÖLLIKER means by the word "selbständig" becomes clear when we read that in every canal spicules project, in which, after treatment with acetic acid, "an der Stelle des in die Flimmercanäle hineinragenden Strahles der genannten Spicula zarte Scheiden leer zurück (bleiben)". These sheaths are, according to KÖLLIKER, perhaps a "Rest von Bildungzellen." However, he adds: "freie Spicula zeigen, der Einwirking der Essigsäure ausgesetzt, keine solche Scheide...." What is meant here with "freie" spicula is not evident. It can hardly mean anything else than isolated spicules. If they be isolated mechanically, the sheath is as obvious as in spicules in situ; if they be isolated chemically, then of course the absence of a sheath is no proof at all.

It is easy enough to repeat KÖLLIKERS experiments, especially in using specimens with thin walls, as e.g. *Leucosolenia*. If a fragment of a wide tube of *L. variabilis* which has been cut open, is spread out in water under the microscope, and carefully treated with acetic acid, the carbonate of lime is seen to be gradually dissolved, and soon the sheaths, with sharply defined outlines, exactly as described by KÖLLIKER, are visible. The sharp outlines are especially clear on spots where the spicules are wholly enclosed by parenchyma; in projecting spicules the conical sheath of spiculine is seen to remain as a homogeneous, extremely thin film; still more striking, perhaps, is the phenomenon if the sponge is stained on the object glass, e. g. with carmalumn (GRÜBLER). The carmalumn, which is generally somewhat acid, causes the calcite to dissolve, and stains the sheath purple; the sharply defined outlines (optical sections) appear dark-purple, while the projecting spicules are faintly purplish. If such preparations are examined in glycerine or in Canada balsam, the spots where the carbonate of lime has been dissolved, or where it is still present, are hardly to be discerned. When we use the polarising microscope, however, the presence of calcite becomes immediately visible. Consequently, there really exists a special layer of organic substance which tightly covers the spicule, which can be isolated with the spiculum, but which cannot be separated from it otherwise than by dissolving the carbonate of lime. Doubtless it is this organic layer which KÖLLIKER called "Scheide". In this sense also MINCHIN uses the word.

The question now arises how far siliceous spicules are likewise enclosed by such organic coats, homologous to the sheaths of the calcareous spicules. We are of opinion that this is actually the case; the delicate organic film which we found covering the spicules of *Tethya* we consider as the homologon of the spicule sheath of calcareous spicules. Such products have been already observed. NOLL (1888 p. 16-17) says: "Noch ist für die Spicula von *Desmacidon Bosei* eines Ueberzugs von organischer Substanz Erwähnung zu thun, ... Hauptsächlich nach Behandlung der Präparate mit einer Höllensteinlösung . . . . weniger deutlich mit Acidum pyrophosphoricum, manchmal auch mit Picrocarmin wurde derselbe sichtbar. Stifte, die isoliert, ohne Ueberzug von verkittendem Spongin, über die Hälfte frei aus dem Schwammgewebe hervorstanden oder auch solche, die ganz frei lagen, waren besonders nach der Silberfärbung gleichmässig mit einem lichtbraunen Ueberzuge versehen, der trotz seiner geringen Dicke doppelte Konturen erkennen liess und die Stifte gleichmässig überdeckte . . . . Die Spicula von *Desmacidon Bosei* besitzen also einen homogenen hautartigen Ueberzug von organischer Substanz, der verschiedene Farbstoffe aufnimmt. Wir wollen ihn als Spicula-Oberhaut bezeichnen . . . ." Although NOLL sees in this coat something else than what KÖLLIKER found in calcareous spicules, we suppose them to be equivalent. Just as KÖLLIKER indicates that his "Scheide", is perhaps a "Rest von Bildungszellen", so NOLL writes that his "Oberhaut" may be "der Rest der die Nadeln bildenden Zellen".

SOLLAS described in the same year (1888) such a sheath, which became perceptible after treatment with hydrofluoric acid. Description and drawing (l. c. p. XLIX, Pl. XLIII, fig. 18), regarding the spicule of "*Dorypleres Dendyi*", leave nothing to be desired as to clearness. "Although at first sight the acid appears to remove all the substance of the spicule except the axial rod, careful observation will show that this is not the case, for a delicate film of organic matter also remains behind; it has the form of a hollow sheath, corresponding in form and position with the outermost boundary of the original spicule; between it and the axial rod the whole of the spicule is completely removed. The spicule thus consists of a central organic axis, surrounded by concentric layers of opal, the outermost of which is invested in a spicule sheath of organic matter or rather of organic matter in intimate association (chemical union?) with silica". Our results regarding the presence of such a sheath in spicules of other species we hope to give in a next publication; for the present we deal only with the spicules of *Tethya*.

It is evident, that if we are right in our conception of the spicule sheath, other coats which sometimes are found surrounding spicules, may not be called sheaths.

F. E. SCHULZE describes in his last paper with his well known accuracy such surroundings from the enormous needles of *Monorhaphis*. We regret not to agree with him in calling this formation "Scheide". As SCHULZE demonstrated it to be not only an investment of each spiculum for itself, but also a means of joining different spicules together, and as this is true for other spicules also, we propose to call it *periapt* [1]). SOLLAS (1880 p. 401 [2]) described a similar coat surrounding spicules from *Isops phlegraei*. BÜTSCHLI found it in *Tethya lyncurium*, a fact which we can confirm. The periapt is composed of connective tissue with conspicuous fibrils and cells; it has therefore, nothing to do with the spicules as such and may be left out of consideration here. Mutatis mutandis, the periapt behaves to the spicule sheath as a perimysium to the sarcolemma.

We have seen already that when the spicopal is dissolved first of all the axial rod appears. The presence of such an organic thread in macroscleres is no more doubted. Although we have no more doubt that the axial thread is normally present, it cannot be denied that in some cases it is wholly or partly absent. We consider such cases, however, to be pathological.

With regard to the shape of the rod BÜTSCHLI (1901 p. 253) writes: "eigenthümlich ist das Querschnittsbild des Fadens, das.... stets deutlich dreieckig erscheint, gleichgültig ob der äussere Umriss des Nadelquerschnitts selbst etwas dreiseitig oder ganz kreisrund ist. Vielfach ist jedoch auf den Nadelquerschnitten zu erkennen, dass der Querschnitt des Achsenfadens sechseckig erscheint, indem die Ecken des dreiseitigen Umrisses regelmässig abgestumpft sind". Consequently, the central thread in such cases, e. g. in *Tethya*, would be a triangular rod and not a cylinder. Whereas already BOWERBANK (1864) apparently had seen something like this in *Geodia*, F. E. SCHULZE (1904) arrived at the conclusion that the axial threads in Hexactinellida are cylindric. In view of this contradiction we thought it necessary to submit the axial thread of *Tethya* to a careful investigation. In order to judge about the shape of a thread in transverse section, we followed the method of BÜTSCHLI by grinding spicules in an agate morter. In the powder, procured in this way, there is found always a sufficient quantity of particles of approximately

[1]) περιάπτω I bind together.

[2]) Not 1890; — apparently this is a misprint in SCHULZE (1904 p. 204), — also not p. 440—441 but 400—401.

cylindrical shape, although that the fracture is irregular. If these pieces are mounted in glycerine,[1]) it is possible, as the spicopal almost entirely disappears from the eye, to judge with certainty in which position the thread is seen, whether oblique or not. Paying attention only to those which are undoubtedly seen in transverse optical section, we found by careful focussing that they were really triangular with the angles cut off. In the second place we studied isolated axial threads. By moving somewhat the coverglass, a quantity of little pieces break off, generally almost transversely. At the same time it can be observed that the little fragments turn over, owing to the movement of the fluid. In this way we saw plenty of them from all sides. With high power (Zeiss, homog. imm.) we found that in this case also the transverse section is triangular. In spite of our astonishment that the axial rod in *Tethya* is triangular we cannot but agree with BÜTSCHLI's observations. We did not see varicosities nor sharp restrictions; normally the isolated rods are perfectly smooth. The diameter remains the same, with exception of the extremities; these are, in uninjured threads, either sharply pointed or rounded off, according to the well known shape of the styli themselves. Deviations of this rule seem to us to be pathological.

Not less strange than the shape appears to be, is the consistency. This may be the reason that previous authors so little agree. Whereas we find, after treatment with hydrofluoric acid, that the free axial threads on one hand are very flexible, so that they can form clews, we see on the other hand that in many cases they break off at once if touched with a needle. We remarked already that they generally break at right angles to the axis. In a certain sense BÜTSCHLI is right, therefore, if he calls them "spröde", but it is by no means the sort of brittleness of for example a thread of glass. The consistency of the axial thread can be best compared with agar-agar. Here a certain flexibility is likewise combined with the property of suddenly breaking. Similar phenomena are known of gels at a certain point of dehydration.

The axial rod in *Tethya* is, taken as a whole, *not* homogeneous. First of all we observe, especially in threads stained with iodine, a double contour. This is easily demonstrated on uninjured, isolated threads as well as in transverse sections; in the triangular figure, mentioned above, the wall is both inside and outside triangular. This wall is comparatively thick — about $^1/_4$ or $^1/_5$ of the total diameter. We may consider the axial thread as a tube, filled with something;

[1]) Refractive index n = 1.4508. Cf. infra.

whereas the wall seems to be homogeneous, the contents are homogeneous or rather granular, apparently of a softer consistency than the more rigid wall. This we conclude from the fact that curved or bent axial threads under the microscope resemble curved or bent indiarubber tubes, filled with a fluid or semi-fluid substance. Solid, flexible cylinders never show such abruptly bent figures.

A remarkable phenomenon is to be seen in broken spicules under the influence of hydrofluoric acid. As stated above, the spicopal is dissolved in a peculiar way, the central canal being hollowed out in the shape of a funnel. If we now only take into consideration the cases where the thread is broken at the same place as the spicopal, we see the thread gradually shrinking somewhat under the influence of the hydrofluoric acid. However, the wall and its contents do not shrink equally. The result is that the contents somewhat pour out beyond the wall. It is not improbable that BÜTSCHLI has seen this; at least his illustration (fig. 24 on pl. XXI) strongly resembles what we observed. But BÜTSCHLI explains it in another way; he believes the thread to be restricted "manschettenförmig".

According to BÜTSCHLI the axial thread consists of a proteid substance. With F. E. SCHULZE we can confirm this in the main points. Boiled in MILLON's fluid the threads in ground spicules turn yellow. This staining is especially distinct in pieces where a part of the thread is lost, and where, consequently, the axis of the spicule is partly uncoloured, partly filled with a yellowish thread. Isolated axial threads or threads partly freed by solution of the spicopal are easily stained with iodine. Treated with nitric acid (25 %) they swell somewhat and acquire a faint yellow colour, which becomes darker by subsequent addition of ammonia. Heated with nitric acid the threads dissolve; likewise in caustic potash. We may conclude, therefore, that the central rod if not wholly, at least partly consists of some proteid. Observed under the polarisation microscope no trace of anisotropy could be seen.

With regard to the styles of *Tethya lyncurium* we thus arrived at the conclusion, that they are composed of an organic axial thread and an organic spicule-sheath, between which elements the spicopal is deposited. We failed in demonstrating any trace of organic (spiculine) lamellae. But still we found, that under special circumstances longitudinal, resp. concentric striae were distinctly seen. We have to look for an explanation of this fact.

In order to avoid a possible misunderstanding or confusion we wish at once to draw attention to the fact, hitherto rather neglected, that one has to distinguish the various *layers* of spicopal from their

limiting plains. We hope to show that the well-known stripes are nothing but the optic sections of such limiting plains, and that they are independent of eventual differences of the layers. It is easy enough to microscopically demonstrate such limiting plains in layers in artificial siliceous gels. If a coat of not yet coagulated siliceous gel is poured out over another one freshly coagulated, and if this is repeated, it becomes evident that the consecutive layers of gel in the beginning do not unite. Only in drying the layers become one mass; still, in section the limiting plains are very conspicuous. This experiment teaches us that in a siliceous gel a lamellar structure can appear, wherein the consecutive layers are separated by visible limits, without interference of another substance e. g. of an organic lamella. We will come back to this fact later.

Several ways are open to us for the study of the spicular structure, but they are not easy. Besides dissolving the spicopal by means of hydrofluoric acid and carefully watching the process, the method of heating has been applied since GRAY (1835) showed that this brings out more distinctly the lamellar structure. IJIMA (1901) was the first to point out the effect of different media. We shall see, that some spicules mounted either in Canada balsam or in glycerine, so widely differ in aspect that on first sight one believes that one has to deal with entirely different sorts of spicules. We thought it necessary, therefore, to begin by determining the refractive index of spicopal of various spicules somewhat more accurately than hitherto done. As far as we know of there exists no other information than given by SOLLAS (1885), who states in general that "the refractive index of sponge-silica is . . . . that of opal or colloidal silica, and not of quartz", and that the spicules come nearest to invisibility when "mounted in chloroform, which possesses a refractive index of 1.449".

In determining the refractive index of the spicules we used the method, since SOLLAS generally used also in mineralogy, viz. to find in what fluid the spicule can no longer be seen. Perhaps there is a still better criterion to make out how much a spicule differs from its medium and in which direction, viz. the appearance of coloured borders. — In order to avoid the effect of fluids wich might influence the amount of water contained in the spicopal, but, on the other hand to demonstrate just this influence, we used fluids which do not mix with water as well as such which were diluted with water. The refractive indices were determined by the refractometer of ABBE, which has the great advantage of enabling us to work with ordinary daylight and to determine any number of indices, between the microscopical work. In spite of the apparatus of PULFRICH being more

accurate we used, therefore, an ABBE, the more so as it turned out to be sufficiently accurate for our purposes.

Among the fluids, not mixable with water, we took advantage of the series liquid paraffin ($n = 1.4759$), petroleum ($n = 1.4568$), benzin ($n = 1.3994$) and petroleum-ether ($n = 1.3780$).

We succeeded by using mixtures of petroleum and benzin in fixing the refractive index of the spicules at 1.4508--1.4510. In order to give an idea of the degree of accuracy that can be attained in this way, we may state that undoubtedly a difference is to be seen between spicules, mounted in a mixture of 20 cc. of petroleum with 3 cc. of benzin ($n = 1.4500$) and mounted in a mixture of 20 cc. of petroleum with 2.5 cc. of benzin ($n = 1.4510$). For a aqueous watery solution we used dilute glycerine, and found with this medium also a complete disappearance at $n = 1.4508$. In addition to these media we studied the influence of air, methylic alcohol, water, potassium acetate, creosote, oil of bergamot, venetian turpentine, oil of red cedar wood, oil of lemon, oil of thyme, firoil, oil of peppermint, oil of cloves (pure or mixed with alcohol), Canada balsam and monobromnaphtaline. Practically however, we used more especially the two fluids mentioned above.

In glycerine with $n = 1.4508$ indeed the spicopal of some spicules disappears completely, and only the axial rod remains visible as a light bluish thread. In all styli, the central thread has a higher refractive index than the spicopal. By careful examination (Zeiss Apochr. 8.0c. 4) in most spicules a light, sharp line may be seen as border, and a system of longitudinal striae between the axial rod and the border. With low power the lamellar structure does not become conspicuous, though the borderline is still visible. Most probably this thin line, which exhibits double contours with high power, represents the organic spicule sheath.

By these experiments it becomes at the same time very evident, that the axial thread can be partly absent; on these spots the light bluish band (the central rod) abruptly ceased. If the glycerine has entered into the central canal, only an indication of the spicopal is visible; if air has entered, of course this is directly visible by the lower refraction. The aspect of styli isolated by means of boiling with hydrochloric acid, either with addition of potassium chlorate or without, or by digestion by means of artificial gastric juice, or by heating with sulphuric acid and potassium-bichromate, fundamentally agree.

Quite another aspect is shown by spicules which have been dried for some days in the presence of anhydrous phosphoric acid. We have

studied spicules which had been dried in this way for some days at an ordinary temperature, and also in VICTOR MEYER's toluol bath at 104°. The heating, however, had no influence on the aspect. The determination of the refractive index is a little less accurate than with ordinary spicules, because on drying the lamellar structure had become somewhat more conspicuous, and therefore the disappearance in glycerine is a little less perfect. Still we could determine in petroleum-benzine the refractive index $n = 1.4052 - 1.4055$; with diluted glycerine the same results were obtained.

If the dried spicules, mounted in glycerine with $n = 1.4055$, and with the border of the coverglass well shut by means of vaseline, are left to themselves, gradually from the outside the refraction is seen to increase, and after one day the spicules become again highly refractive. If they are now as much as possible separated from adhering glycerine and transferred into glycerine of $n = 1.4508$, it is seen that they disappear in this medium, and consequently have absorbed again their original quantity of water. Spicules, which have been dried by P_2O_5 and are exposed to the air afterwards, behave in the same way. On the other hand we have examined the behaviour of spicules which, after being isolated and washed, were not dried in the air, but immediately after removing the adjacent water, were mounted in glycerine with $n = 1.4508$. We could indeed see a very slight difference with spicules which had been dried in the air at an ordinary temperature, more than corresponded with possibly adherent water, but too small to be measured. However, it appears that, even at the ordinary temperature, the spicopal gives off some water.

Consequently we have demonstrated that the spicopal is a form of hydrated siliceous acid, which may give off water in an atmosphere, dried by P_2O_5, which diminishes hereby in refractive index, and which may again absorb the original quantity of water by immersion in a watery solution or by exposure to moist air. The spicopal also behaves with regard to the absorptive power for water just as a gel, as has been shown by the well known and intricate researches of VAN BEMMELEN.

(To be continued).

Mathematics. — *"On pencils of algebraic surfaces."* By Prof. JAN DE VRIES.

1. Let a pencil (F^n) of surfaces F^n of order n be given, intersecting in the base-curve σ.

The principal tangents in a point S of σ to the surfaces of (F^n) form a cubic cone having the tangent s to σ for edge.

For, if (F^n) is indicated by

$$a_x^n + \lambda b_x^n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and if y_k are the coordinates of S, the substitution $x_k = y_k + \varrho z_k$ furnishes in connection with $a_y^n = 0$ and $b_y^n = 0$ for a point Z on a principal tangent the conditions

$$a_y^{n-1} a_z + \lambda b_y^{n-1} b_z = 0 \quad , \quad a_y^{n-2} a_z^2 + \lambda b_y^{n-2} b_z^2 = 0 ,$$

so that the locus of the principal tangents touching in S has as equation

$$a_y^{n-1} b_y^{n-2} a_z b_z^2 - a_y^{n-2} b_y^{n-1} a_z^2 b_z = 0 \quad . \quad . \quad . \quad . \quad (2)$$

If Z is a fixed point, Y a variable one, this equation represents a surface of order $(2n - 3)$ determining on σ the points S which are points of contact of principal tangents through Z.

The principal tangents in points of the base-curve form a congruence of order $n^2(2n - 3)$ *and of class* $3n^2$.

The inflexional tangents of a pencil of plane curves c^n enveloping a curve of class $3n(n - 2)$, *the complex of rays of the principal tangents* of F^n is of order $3n(n - 2)$.

2. A principal tangent t_3 in S becomes four-pointed tangent t_4, if for a point Z lying on it the relation $a_y^{n-3} a_z^3 + \lambda b_y^{n-3} b_z^3 = 0$ holds good. So the tangents t_4 touching in S belong to the biquadratic cone

$$a_y^{n-1} b_y^{n-3} a_z b_z^3 - a_y^{n-3} b_y^{n-1} a_z^3 b_z = 0 \quad . \quad . \quad . \quad . \quad (3$$

As the cones (2) and (3) have the tangent s represented by

$$a_y^{n-1} a_z = 0 \quad , \quad b_y^{n-1} b_z = 0 \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in common, the point S will be the point of contact of *eleven four-pointed tangents.*

For $a_y^n = 0$ and $b_y^n = 0$ the equations (2) and (3) represent the figure formed by the surface of tangents (s) of σ and the scroll τ_4 of the right lines t_4 having their points of contact on σ.

To determine the order of τ_4 I search for the number of points

of intersection of the indicated figure with the right line $z_3 = 0$, $z_4 = 0$. Substitution in (2) and in (3) and elimination of z_1 and z_2 gives an equation containing the coefficients of (2) and (3) successively in the orders 4 and 3. Hence the resultant in the coordinates y is of order $4(2n-3)+3(2n-4)$ or $14n-24$.

So the number of points of intersection is $2n^2(7n-12)$.

Applying the same treatment to (4) I find for the order of (s) the well-known number $2n^2(n-1)$.

The four-pointed tangents having their points of contact on the base-curve σ form a scroll of order $2n^2(6n-11)$, on which σ is elevenfold.

For $n=3$ this scroll passes into the locus of the right lines on the surfaces of a pencil (F^3), thus into the scroll of the trisecants of σ^9 which is of order 42. For $2n^2(6n-11)$ we now find 126, corresponding to the fact, that each trisecant does duty as right line t_4 for three points S.

On each surface F^n the points of contact P_4 of four-pointed tangents t_4 form a curve of order $n(11n-24)$. As σ is evidently an elevenfold curve of the locus of the points of contact P_4 belonging to the F^n of the pencil this locus has in common with every F^n a locus of order $n(11n-24)+11n^2$.

The points of contact of four-pointed tangents form a surface of order $2(11n-12)$.

For $n=3$ we find the scroll of trisecants of order 42.

3. As the tangents t_3 passing through a point Z form a cone of order $3n(n-2)$ and two principal tangents have their point of contact in Z the locus of the points of contact P_3 of the right lines t_3 containing Z is a curve of order $3n(n-2)+2$ with double point in Z.

Each of those right lines t_3 cuts the surface F^n osculated by it in $(n-3)$ points Q more. The locus of the points Q is a curve of order $3n(n-2)(n-3)+(n^2+2)(n-3)$ or $2(n-3)(n-1)(2n-1)$. For, through Z pass $(n^2+2)(n-3)$ tangents t_3, osculating the surface indicated by Z in an other point [1]).

To find the number ε of the coincidences of P_3 with Q, I make use of the well-known formula

$$\varepsilon = p + q - g,$$

which appears when the pairs of points P, Q are projected by a pencil of planes. Each point P belonging to $(n-3)$ pairs we have

[1]) See CREMONA—CURTZE, Oberflächen, p. 66.

$p = (3n^2 - 6n + 2)(n - 3)$. Farther more $q = 2(n - 3)(n - 1)(2n - 1)$, whilst the number of right lines PQ resting on an axis is of course equal to $3n(n - 2)(n - 3)$. So $\varepsilon = 2(n - 3)(2n^2 - 3n + 2)$; this is also the number of four-pointed tangents through a given point a.

The number of right lines t_4 in a given plane is equal to the number of points of undulation on the curves c^n of a pencil; this number I have determined in a preceding paper [1]).

The four-pointed tangents form a congruence of order $2(n - 3)(2n^2 - 3n + 2)$ *and of class* $\frac{9}{2}(n-3)(n^3+n^2-8n+4)$.

4. If we wish to apply the above-mentioned formula of coincidence to the pairs of points of intersection Q, Q' on the right lines t_3 through Z we have to substitute $p = q = 2(n - 3)(n - 1)(2n - 1)(n - 4)$ and $g = 3n(n - 2)(n - 3)(n - 4)$. For each point Q belongs to $(n-4)$ pairs and each right line t_3 bears $(n - 3)(n - 4)$ pairs. We then find $\varepsilon = (n - 3)(n - 4)(5n^2 - 6n + 4)$, i. e. the number of tangents $t_{3,2}$ through the point Z.

In the above-mentioned paper I have determined the number of right lines having with a curve of a pencil (c^n) a three-pointed and at the same time a two-pointed contact.

The two-three-pointed tangents form a congruence of order $(n - 3)(n - 4)(5n^2 - 6n + 4)$ *and of class*

$$\frac{1}{2}(n - 4)(n - 3)^2(10n^4 + 35n^3 - 21n^2 - 80n + 20).$$

5. Each principal tangent t_3 having its point of osculation in a point S of the base-curve bears still $(n - 3)$ points of intersection Q with the surface F^n osculated by it. As S is point of contact of 11 four-pointed tangents the locus of the points Q will have an eleven-fold point in S. As an arbitrary plane through S evidently contains $3(n - 3)$ points Q (§ 1) the order of the curve (Q) is equal to $(3n + 2)$.

When applying the formula $\varepsilon = p + q - g$ to the pairs Q, Q' which the cubic cone with vertex S bears, we have to put $p = q = (3n + 2)(n - 4)$ and $g = 3(n - 3)(n - 4)$. Then we get $\varepsilon = (n - 4)(3n + 13)$. So this is the number of tangents $t_{3,2}$ for which the point of osculation lies in S.

In other words, σ is an $(n - 4)(3n + 13)$-fold curve on the locus $[R_3]$ of the *points of osculation* of tangents $t_{3,2}$ to surfaces of (F^n). Now the points of osculation of the right lines $t_{3,2}$ of an F^n form a

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[1]) "On linear systems of algebraic plane curves", (Proceedings, April 22, 1905.)

curve of order $n(n-4)(3n^2+5n-24)$ [1]). So $[R_3]$ has in common with an F^n of the pencil a curve of order

$$n(n-4)(3n^2+5n-24)+n^2(n-4)(3n+13)=n(n-4)(6n^2+18n-24).$$

The points of osculation of the three-two-pointed tangents of (F^n) *form a surface of order* $6(n-1)(n-4)(n+4)$.

6. To determine the order of the cone formed by the double tangents of (F^n) of which a point of contact in S lies on the base-curve σ, we notice that the tangent s in S to σ is intersected by a pencil in an involution of order $(n-2)$. Its $2(n-3)$ double points are points of contact of double tangents touching in S too. So s is a $2(n-3)$-fold edge of the indicated cone.

In each plane ϕ through s we can draw out of S $n(n-1)-6$ tangents to the curve of intersection of ϕ with the surface F^n touching ϕ in S. From this ensues that the indicated cone is of order $(n-3)(n+4)$.

The locus of the second points of contact R_2 of the edges of this cone has evidently in S an elevenfold point, where it is touched by the eleven right lines t_4. So the curve (R_2) is of order

$$(n-3)(n+4)+11=n^2+n-1.$$

Every edge of the cone intersects the surface doubly touched by it in $(n-4)$ points V more. The locus of these points passes $(n-4)(3n+13)$ times through S, where it is touched by the right lines $t_{3,2}$ osculating in S. As each plane through S bears moreover $(n^2+n-12)(n-4)$ points V, the curve (V) is of order

$$(n-4)(n^2+4n+1).$$

Now the number of coincidences of R_2 with V can be determined again by means of the formula $\varepsilon=p+q-g$. We find

$$\varepsilon=(n^2+n-1)(n-4)+(n-4)(n^2+4n+1)-(n-3)(n+4)(n-4)=$$
$$=(n-4)(n^2+4n+12).$$

This is the nnmber of tangents $t_{3,2}$, of which the *point of contact* lies in S, thus at the same time the multiplicity of the base-curve on the surface $[R_2]$ of the points of contact of surfaces of pencils with right lines $t_{3,2}$. Taking into consideration, that the points of contact R_2 form on the surface F^n a curve of order

$$n(n-2)(n-4)(n^2+2n+12)$$

we find that $[R_2]$ has with F^n an intersection of order

$$n(n-2)(n-4)(n^2+2n+12)+n^2(n-4)(n^2+4n+12).$$

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[1]) See inter alia my paper: "Some characteristic numbers of an algebraic surface." Proceedings, April 22nd, 1905).

The points of contact of the three-pointed tangents of (F^n) *form a surface of order* $2(n-4)(n^3+2n^2+10n-12)$.

7. Through the tangent s in S to σ we can make to pass four tangent planes to the cubic cone of the principal tangents (§ 1). So S is a parabolic point on four surfaces of the pencil. Therefore σ is a fourfold curve on the locus of the parabolic points.

As the parabolic points of an F^n lie on a curve of order $4n(n-2)$ the locus under consideration is cut by each of the surfaces F^n in a curve of order $4n(n-2)+4n^2=8n(n-1)$.

The locus of the parabolic points of the surfaces of a pencil (F^n) *is a surface of order* $8(n-1)$.

Chemistry. — "*On the shape of the plaitpoint curve for mixtures of normal substances.*" (Second communication). By J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

1. In a previous paper [1]), starting from VAN DER WAALS' equation of state, in which b is assumed to be independent of v and T, I have found for the equation of the *spinodal curves* at successive temperatures (l. c. p. 690):

$$RT=\frac{2}{v^2}\left[x(1-x)\theta^2+a(v-b)^2\right], \quad . \quad . \quad . \quad . \quad (1)$$

and for that of the plaitpoint curve in its v, x projection (l.c. p. 695):

$$x(1-x)\theta^2\left[(1-2x)v-3x(1-x)\beta\right]+\sqrt{a}(v-b)^2\left[3x(1-x)\theta(\theta-\beta\sqrt{a})+\right.$$
$$\left.+a(v-b)(v-3b)\right]=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this $\theta=\pi+\alpha(v-b)$, $\pi=b_1\sqrt{a_2}-b_2\sqrt{a_1}$, $\alpha=\sqrt{a_2}-\sqrt{a_1}$, and $\beta=b_2-b_1$.

The equations (1) and (2) hold for the so-called *symmetrical* case, where not only $b_{12}=\frac{1}{2}(b_1+b_2)$ is assumed, but also $a_{12}=\sqrt{a_1a_2}$. These hypotheses lead to:

$$b=(1-x)b_1+xb_2 \quad ; \quad a=[(1-x)\sqrt{a_1}+x\sqrt{a_2}]^2.$$

The equation (1) had been given already before by VAN DER WAALS in implicit form [2]), for after some reduction his general equation

[1]) These Proc. April 22, 1905, p. 646—657.

[2]) Cont. II, p. 45; Arch. Néerl. **24**, p. 52 (1891).

passes really into (1) after substitution of the values of $\frac{da}{dx}$, $\frac{db}{dx}$, $\frac{d^2a}{dx^2}$ and $\frac{d^2b}{dx^2}$ in accordance with the above hypotheses.

But the equation (2) may be said to have been derived here for the first time in the above simple form. It is true that VAN DER WAALS gave a differential equation of this curve [1]), and derived an *approximate* rule for its shape [2]), but he did not arrive at a general final expression. Nor has KORTEWEG arrived at it in his very important papers: "Sur les points de plissement" and "La théorie générale des plis, etc." [3]) In his final equation (73) (l.c. p. 361) there occur, besides T, still several functions $\varphi(v)$, $\zeta(v)$, $\psi(v)$ and $\chi(v)$, which have been given respectively by the equations (37), (38), (40) and (74) (l. c. p. 350 and 361). KORTEWEG's equation is one of the 9th degree with respect to v, but it is easy to see that it may be reduced to one of the 8th degree (l. c. p. 361). It appears from our derivation that this degree may be reduced to the 4th. In a later paper [4]) KORTEWEG confines himself to a full discussion of the plaitpoints in the neighbourhood of the borders of the ψ-surface.

I think that one of the reasons for failure in this direction is due to the intricate form of the differential equation of the plaitpoint curve, when we use the ψ-function. The ζ-function on the other hand leads to simpler expressions. Already the differential equation for the spinodal line at given temperature, viz. $\left(\frac{\partial^2\zeta}{\partial x^2}\right)_{p,T} = 0$ or $\left(\frac{\partial\mu_1}{\partial x}\right)_{p,T} = 0$, is much simpler than the corresponding expression in ψ. And to get the plaitpoint curve, we have only to combine $\left(\frac{\partial\mu_1}{\partial x}\right)_{p,T} = 0$ with $\left(\frac{\partial^2\mu_1}{\partial x^2}\right)_{p,T} = 0$.

2. We shall now examine the shape of the curves given by (1) and (2) more closely, and specially for the case that $\beta = 0$, i. e. $b_1 = b_2 = b$. The calculations are rendered very simple in this way, and it is obvious from the adjoined figs. 1—4, that when b_1 is not $= b_2$, so β not $= 0$, the results will be modified only quantitatively, but by no means qualitatively. We shall come back to this in a following paper.

[1]) Verslagen Kon. Akad. Amsterdam, 4, p. 20—30 en 82—93 (1896).
[2]) Id. 6, p. 279—303 (1898).
[3]) Arch. Néerl. 24, p. 57—98 en 295—368 (1891).
[4]) These Proc. Jan. 31, 1903, p. 445.

As $\theta = \pi + \alpha(v-b) = \alpha v - \beta\sqrt{a}$ passes into αv for $\beta = 0$, we may write for (1):

$$RT = \frac{2}{v^3}\left[x(1-x)\alpha^2 v^3 + a(v-b)^3\right], \quad \ldots \quad (1a)$$

and (2) is reduced to:

$$x(1-x)\alpha^3 v^3[(1-2x)v] + \sqrt{a}(v-b)^2\left[3x(1-x)\alpha^2 v^2 + a(v-b)(v-3b)\right] = 0. \quad (2a)$$

Let us put these equations into a more homogeneous form.

As $a = [\sqrt{a_1} + x(\sqrt{a_2} - \sqrt{a_1})]^2 = (\sqrt{a_1} + x\alpha)^2$, we may write for (1a):

$$RT = \frac{2}{v}\left[x(1-x)\alpha^2 + (\sqrt{a_1} + x\alpha)^2\left(1 - \frac{b}{v}\right)^2\right]$$

$$= \frac{2\alpha^2}{v}\left[x(1-x) + \left(\frac{\sqrt{a_1}}{\alpha} + x\right)^2\left(1 - \frac{b}{v}\right)^2\right].$$

If we now put:

$$\frac{\sqrt{a_1}}{\alpha} = \varphi \quad ; \quad \frac{b}{v} = \omega,$$

this last equation becomes:

$$RT = \frac{2\alpha^2}{b}\omega\left[x(1-x) + (\varphi + x)^2(1-\omega)^2\right].$$

Let us now introduce the "third" critical temperature T_0. This temperature is the plaitpoint temperature at $v = b$, i. e. that at which the limiting curve lying in the limiting plane $v = b$ (see fig. 1 of my previous paper cited above) reaches its maximum, and is represented by ($\omega = 1$):

$$RT_0 = x_c(1 - x_c)\frac{2\alpha^2}{b}.$$

But as in the case $b_1 = b_2$ for x_c the value $^1/_2$ is found (the maximum of the now parabolic curve), we get:

$$RT_0 = \frac{^1/_2\,\alpha^2}{b}.$$

Our equation for RT becomes therefore:

$$RT = 4\,RT_0\,\omega\left[x(1-x) + (\varphi + x)^2(1-\omega)^2\right].$$

And if henceforth all temperatures are expressed in multiples of T_0, we have finally, putting

$$\frac{T}{T_0} = \tau,$$

$$\tau = 4\omega\left[x(1-x) + (\varphi + x)^2(1-\omega)^2\right] \quad (1b)$$

In this simple form the equation is very suitable for calculating successive spinodal curves. It is of the *second* degree with respect to x, of the *third* degree with respect to ω. For a given value of τ we have therefore only to put successively $\omega = 1,\ 0{,}9,\ 0{,}8$ etc. down to 0, and then we find the corresponding values of x by solution of ordinary quadratic equations.

The equation $(2a)$ becomes after division by $x(1-x)\,a^2v^4$:

$$(1-2x)+\frac{\sqrt{a}}{a}\left(1-\frac{b}{v}\right)^2\left[3+\frac{a/a^2\,(1-b/v)\,(1-3b/v)}{x\,(1-x)}\right]=0,$$

i. e. as $\dfrac{\sqrt{a}}{a}=\dfrac{\sqrt{a_1}}{a}+x=\varphi+x$

$$(1-2x)+(\varphi+x)(1-\omega)^2\left[3+\frac{(\varphi+x)^2}{x\,(1-x)}\,(1-\omega)\,(1-3\omega)\right]=0\ldots(2b)$$

This equation of the plaitpoint curve is of the *third* degree with respect to x, of the *fourth* degree with respect to ω.

3. Before discussing the equations $(1b)$ and $(2b)$ more fully, we shall first derive a few relations between T_0, T_1 and T_2.

As $RT_0=\dfrac{^1/_8\,a^2}{b}$ (see above) and $RT_1=\dfrac{8}{27}\dfrac{a_1}{b}$, we find immediately:

$$\frac{T_1}{T_0}=\frac{16}{27}\frac{a_1}{a^2}=\frac{16}{27}\varphi^2.$$

From this follows that for values of $\varphi < {}^3/_4\sqrt{3}\ (=1{,}30)$ T_1 will be $< T_0$; i. e. the lower critical temperature of the two components will then be lower than the critical temperature of mixing of the two liquid phases at $v=b$.

As $\varphi=\dfrac{\sqrt{a_1}}{\sqrt{a_2}-\sqrt{a_1}}$, so $\dfrac{1}{\varphi}=\dfrac{\sqrt{a_2}}{\sqrt{a_1}}-1$, and we have evidently:

$$\frac{T_2}{T_1}=\left(1+\frac{1}{\varphi}\right)^2.$$

For $\varphi=0$ is $T_2=\infty\times T_1$; for $\varphi=\infty$ is $T_2=T_1$. For $\varphi={}^3/_4\sqrt{3}$ (see above) $T_2/T_1=(1+{}^4/_9\sqrt{3})^2={}^1/_{27}(43+24\sqrt{3})=3{,}13$.

It will also prove important to know the amount of the *pressure* for all points of the spinodal curves. For this purpose we reduce the equation:

$$p=\frac{RT}{v-b}-\frac{a}{v^2}$$

to

$$p = \frac{RT}{v(1-b/v)} - \frac{(\sqrt{a_1}+x\alpha)^2}{v^2} = \frac{RT}{v(1-\omega)} - \frac{\alpha^2}{v^2}(\varphi + x)^2.$$

This becomes on account of $\alpha^2 = 2bRT_0$ (see above) and $T/T_0 = \tau$:

$$p = \frac{RT_0}{b}\omega\left\{\frac{\tau}{1-\omega} - 2\omega(\varphi + x)^2\right\}.$$

Let us express p in the critical pressure p_1. (As, namely, the pressure p_0 corresponding to T_0 ($v = b$) is evidently $= \infty$, p cannot be expressed in p_0). As $p_1 = 1/8 \frac{RT_1}{b}$ and $\frac{T_1}{T_0} = \frac{16}{27}\varphi^2$, $p_1 = \frac{2}{27}\frac{RT_0}{b}\varphi^2$, hence — when we put

$$\frac{p}{p_1} = \pi:$$

$$\pi = \frac{27}{2}\frac{\omega}{\varphi^2}\left[\frac{\tau}{1-\omega} - 2\omega(\varphi + x)^2\right] \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This equation may be used, when τ is already known from (1b). If this value is, however, substituted, we get:

$$\pi = \frac{27}{\varphi^2}\frac{\omega^2}{1-\omega}\left[2x(1-x) + 2(\varphi+x)^2(1-\omega)^2 - (\varphi+x)^2(1-\omega)\right],$$

i. e.

$$\pi = \frac{27}{\varphi^2}\frac{\omega^2}{1-\omega}\left[2x(1-x) + (\varphi+x)^2(1-\omega)(1-2\omega)\right] \quad . \quad . \quad . \quad . \quad (3a)$$

4. Better than descriptions and calculations the adjoined figures 1—4 represent the different relations which may present themselves in the discussion of (1b) and (2b), combined with 3 or (3a). We shall therefore confine ourselves in the following to what is strictly indispensable.

Two principal types occur, according as $\varphi < 1{,}43$ or $> 1{,}43$. Fig. 1 with $\varphi = 1$ is a representative of the one type, fig. 2 with $\varphi = 2$ of the other. The transition case $\varphi = 1{,}43$ is represented in fig. 4.

a. *Description of the case* $\varphi = 1$ (fig. 1 and 1a).

There are *two* plaitpoint curves, one of which extending from C_0 to C_2, the other from C_1 to A. The latter, however, may only be realized down to a point between C_1 and R_1, where it is touched by the spinodal line $\tau = 0{,}63$ [1]).

[1]) See KORTEWEG, l. c. p. 305 (fig. 12) and plate F_1 to F_5. (The plaitpoint has already disappeared in the limiting line $v = b$ in our case). R_1 is a so-called point de plissement double hétérogène. Cf. also VAN DER WAALS, These Proc. V, 310, Oct. 25, 1902.

Beyond the point R_1 the temperature, and with it also the pressure, decreases, as is to be seen from the succession of the different spinodal curves, so that in the p,T-diagram (fig. 1a) the plaitpoint curve C_1R_1A shows a cusp at R_1, and begins to run back.

It is known that this case is realized with mixtures of C_2H_6 and CH_3OH, ether and water (KUENEN), etc. It is the *principal type I*, as I have fully described it in one of my two preceding papers [1]).

Remarkable and quite unexpected is the fact that this type may be realized for mixtures of *normal* substances. It was formerly believed that such deviating plaitpoint curves were only possible when at least one of the two substances is anomalous. This, however, seems not to be the case; more and more the conviction gains ground with me that the anomaly of one or of both components only *accentuates* the phenomena *sharper* or brings them into *attainable* regions of temperature.

It is also striking in fig. 1a, that the curve C_0C_2 has the same appearance, viz. with an inflection in the middle part, as the typical curve as observed by KUENEN for $C_2H_6 + CH_3OH$ (see fig. 1 of my just cited paper). Only in our case there is not yet a pronounced maximum and minimum, as with the mixtures of C_2H_6 with the strongly *anomalous* substance CH_3OH.

The type of fig. 1 occurs for comparatively small values of φ. According to the equation given in § 3 the proportion $T_2/T_1 = 4$ corresponds with $\varphi = 1$. The critical temperatures of the two components must, therefore, lie comparatively far apart.

As $T_1/T_0 = \frac{16}{27}$, T_0 is considerably higher than T_1. If we put $T_0 = 1$, as has been done in the figure, then $T_1 = 0{,}59$ and $T_2 = 2{,}37$.

b. *Some mathematical and numerical details.*

The plaitpoint curve C_0C_2 touches the line $x = {}^1/_2$ in C_0, the curve AC_1 touches the line $x = 0$ in A. Moreover the curve C_0C_2 touches the line $x = {}^1/_2$ once more in D, and it does so at $\omega = {}^2/_3$ ($v = 1{,}5\,b$). In C_1 and C_2 no contact takes place.

When φ becomes < 1, and approaches to 0 (T_2/T_1 then becomes larger and larger and approaches to ∞), then the curve C_1A approaches the straight line $x = 0$ more and more and the curve C_0C_2 the dotted curve in the figure, which continues to present a clearly

[1]) These Proc. VII p. 636—638, April 22, 1905.

pronounced inflection point up to the last [1]). For values of $\varphi > 1$, the curve C_0C_2 lies partially on the left of the curve $x = {}^1/_2$, and the point of contact at D passes into two points of intersection.

By an approximate solution of (2b) and substitution in (1^b) and (3) of the values found, the following points of the two plaitpoint curves are calculated. (The other values of ω or x are either imaginary or do not satisfy).

$\varphi = 1$

Curve C_0C_2						*Curve* C_1A							
$x =$	0,5	0,6	0,7	0,8	0,9	1	$\omega =$	0,33	0,4	0,5[1])	0,6	0,7	0,8	0,9	1
$\omega =$	1	0,49	0,43	0,39	0,36	0,33	$x =$	0	0,021	0,041	0,042	0,023	0,010	0,0017	0
$\tau =$	1	1,78	1,98	2,13	2,26	2,37	$\tau =$	0,59	0,63	0,62	0,51	0,33	0,16	0,042	0
$\pi =$	∞	6,44	5,75	5,05	4,51	4	$\pi =$	1	1,15	1,08	0	−3,09	−8,64	−16,9	−27

It is seen that the pressure begins to be *negative* for points in the neighbourhood of A. This is not remarkable; also for a simple substance the points of inflection in the ideal isotherms reach to within the region of the negative pressures. Though the pressures in some points on the *spinodal* curve are negative, this is no reason why those on the connodal curves should be so.

The limits of the region of negative pressures on the spinodal curves may be easily fixed (see the dotted curves in fig. 1) by solution of the equation (see (3a))

$$2x(1-x) = (\varphi + x)^2 (1-\omega)(2\omega - 1).$$

If we put here $(1-\omega)(2\omega-1) = \theta$, we find:

$$x = \frac{(1 - \varphi\theta) \pm \sqrt{1 - 2\varphi(\varphi+1)\theta}}{2+\theta}.$$

In this way we calculate for $\varphi = 1$:

$\omega =$	1	0,9	0,8	0,7	0,6	0,5
$x =$	0	0,04[5]	0,07	0,07	0,04[5]	0
	1	0,84	0,75[5]	0,75[5]	0,84	1.

That π approaches to -27 for $x = 0$, $\omega = 1$, $\tau = 0$ follows immediately from (3). For as $\frac{\tau}{1-\omega}$ approaches to 0, as we shall prove presently, $\pi = \frac{27}{2}\frac{1}{\varphi^2}\left(-2\varphi^2\right) = -27$, independent of the value of φ.

[1]) For this plaitpoint curve $\varphi = 0$ the following points are easily calculated:

$\omega =$	0,9	0,8	0,7	0,6	0,5	0,4
$x =$	0,507	0,528	0,567	0,623	0,712	0,853

The equation (2b), viz., passes then into the following quadratic equation in $x\omega$:

$$x^2\omega^2(9 - 10\omega + 3\omega^2) - 3x\omega(2-\omega) + 1 = 0.$$

The other value for x is always > 1.

[2]) The maximum lies at $\omega = 0{,}54$; x is then about $= 0{,}043$.

In this we must notice that in the immediate neighbourhood of the point A, π increases with the utmost rapidity from -27 to $+\infty$, when we pass the above considered border curve; *in* the point A itself this transition takes of course place suddenly. For when $\omega = 1$, π approaches to $\frac{27}{\varphi^3}\frac{2x(1-x)}{1-\omega} = \infty$, according to (3*a*), *except* in the case that x is exactly $= 0$, when (see further) $\frac{2x(1-x)}{1-\omega} = 0$, the following term yielding then the finite value -27. This follows also from the figure, because the border curve, which separates positive from negative pressures, passes *through* the point A.

That on the plaitpoint curve the expressions $\frac{x}{1-\omega}$ and $\frac{\tau}{1-\omega}$ approach to 0 for $x = 0$, $\omega = 1$, $\tau = 0$ at A, follows from (2*b*). For putting $x = \Delta$ and $1 - \omega = \delta$, we get:

$$1 + \varphi\delta^2\left(3 - 2\frac{\varphi^2}{\Delta}\delta\right) = 0,$$

or as $3\varphi\delta^2$ may be neutralized by 1, $1 - 2\varphi^2\frac{\delta^3}{\Delta} = 0$, from which follows, that at the point A $\underline{\frac{\Delta}{\delta^3} = 2\varphi^2}$, so remains finite. So Δ is of the order δ^3, so that $\frac{x}{1-\omega} = \frac{\Delta}{\delta}$ really approaches to 0 at A. From this follows also the *contact*. And as according to (1*b*) τ approaches to $4(\Delta + \varphi^2\delta^2) = \underline{4\varphi^2\delta^2}$ (Δ being of the order δ^3) for $x = 0$, $\omega = 1$, $\frac{\tau}{1-\omega}$ approaches to 0 at A.

In the same way the plaitpoint curve C_0C_2 touches the line $x = {}^1/_2$ for $x = {}^1/_2$, $\omega = 1$. For, for $x = {}^1/_2(1 + \Delta)$, $\omega = 1 - \delta$ equation (2*b*) becomes:

$$-\Delta + (\varphi + {}^1/_2)\delta^2\left[3 - 8(\varphi + {}^1/_2)^2\delta\right] = 0,$$

which appoaches to $-\Delta + 3(\varphi + {}^1/_2)\delta^2 = 0$, yielding $\underline{\frac{\Delta}{\delta^2} = 3(\varphi + {}^1/_2)}$, so again finite. So Δ is now of the order δ^2, and so $\frac{\Delta}{\delta}$ again $= 0$, which proves the contact at C_0.

I call attention to the fact, that on account of the small values of Δ a large portion of the curve C_0C_2 from C_0 as far as beyond the point D may be calculated very accurately, by writing for (2*b*) ($\varphi = 1$):

$$-\Delta + {}^{3}/_{2}(1-\omega)^{2}\left[3 - 9(1-\omega)(3\omega-1)\right],$$

so that $$\Delta = {}^{9}/_{2}(1-\omega)^{2}\left[1 - 3(1-\omega)(3\omega-1)\right],$$

From this follows e.g. for $\omega = 0{,}9,\ 0{,}8,\ 0{,}7,\ 0{,}6$ resp., for Δ 0,022, 0,029, 0,004, 0,029.

The contact at D. If we put in $(2b)$ $x = {}^{1}/_{2}$, then $1 - 2x = 0$, and hence:

$$(\varphi + {}^{1}/_{2})(1-\omega)^{2}\left[3 + 4(\varphi + {}^{1}/_{2})^{2}(1-\omega)(1-3\omega)\right] = 0\,.$$

This yields besides $\omega = 1$ (the point C_0), also:

$$(1-\omega)(3\omega-1) = \frac{3}{(2\varphi+1)^{2}},$$

hence:

$$\omega = {}^{2}/_{3} \pm {}^{1}/_{3}\sqrt{1 - \frac{9}{(2\varphi+1)^{2}}}.$$

For $\varphi = 1$ this yields two equal roots $\omega = {}^{2}/_{3}$, which proves the contact at D. For $\varphi < 1$ the roots become imaginary, so that then C_0C_2 no longer cuts the line $x = {}^{1}/_{2}$, but keeps continually on its right, whereas for $\varphi > 1$ two points of intersection are always found. So is e. g. for $\varphi = 2$ $\omega = {}^{14}/_{15}$ (close to C_0) and $\omega = {}^{2}/_{5}$ (lying on the other branch between C_1 and C_2 (see fig. 2)).

In order to facilitate the tracing of the different *spinodal* lines, it is to be recommended to fix the limiting values of τ for $x = 0$, $x = 1$, $\omega = 1$, $\omega = {}^{1}/_{3}$. Also for $x = {}^{1}/_{2}$ it is easy to calculate τ. From $(1b)$ follows e.g. for $x = 0$, $\varphi = 1$:

$$\tau = 4\omega(1-\omega)^{2}.$$

This yields:

$\omega = 1$	0,9	0,8	0,7	0,6	0,5	0,4	0,333	0,3	0,2	0,1
$\tau = 0$	0,036	0,128	0,252	0,384	0,50	0,576	0,593	0,588	0,512	0,324

For $x = 1$ these values become simply 4 times larger, $(\varphi + x)^{2}$ then being $= 4$.

For $x = {}^{1}/_{2}$ we get,

$$\tau = \omega\{1 + 9(1-\omega)^{2}\},$$

yielding:

$\omega = 1$	0,95	0,9	0,8	0,7	0,6	0,5	0,4	0,33	0,3
$\tau = 1$	0,971	0,981	1,09	1,27	1,46	1,62⁵	1,70	1,67	1,62.

For $\omega = 1$ we get simply:

$$\tau = 4x(1-x),$$

from which follows:

$x = 0$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\tau = 0$	0,36	0,64	0,84	0,96	1	0,96	0,84	0,64	0,36	0.

Finally we get for $\omega = {}^1/_3$:

$$\tau = {}^4/_3 \left[x\,(1-x) + {}^4/_9\,(x+1)^2 \right],$$

yielding:

$x =$	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\tau =$	0,593	0,837	1,07	1,28	1,48	1,67	1,84	1,99	2,13	2,26	2,37.

It appears from the diagram (see also above for $x = {}^1/_3$), that the temperature from C_0 to C_2 is not continually ascending, but that it shows a *minimum* very near C_0. This causes the spinodal line $\tau = 1$ *not* to pass through C_0, but to remain under it. The point C_0, where τ is also $= 1$, is an isolated point belonging to that line. Just beyond C_0 the two branches of one and the same spinodal line intersect in a double point; beyond that place the course is normal; between C_0 and this intersection the spinodal line has two separate branches, one of which encloses the point C_0. Now the question arises, whether this will be the case for every value of φ. If we solve x from (1*b*), we get:

$$x^2\,(2\omega - \omega^2) - x\left(1 + 2\varphi\,(1-\omega)^2\right) + \left(\frac{\tau}{4\omega} - \varphi^2\,(1-\omega)^2\right) = 0\,.$$

This gives for x two roots of the same value for given values of τ and ω, when

$$4\,\varphi\,(\varphi+1)\,(1-\omega)^2 + 1 - \tau\,(2-\omega) = 0.$$

The value of x is then:

$$x = \frac{{}^1/_2 + \varphi(1-\omega)^2}{\omega(2-\omega)}.$$

Now it follows from the value of the above given discriminant, that it becomes $= 0$ for *two* values of ω. So two branches of a spinodal line intersect, when those values of ω become the same. From

$$4\varphi\,(\varphi+1)\,\omega^2 - \omega\left(8\varphi\,(\varphi+1) - \tau\right) + \left(4\varphi\,(\varphi+1) + 1 - 2\tau\right) = 0$$

follows, that ω has two roots of the same value, when

$$\frac{\tau^2}{1-\tau} = 16\varphi\,(\varphi+1) \quad ; \quad \omega = 1 - {}^1/_8\,\frac{\tau}{\varphi\,(\varphi+1)}.$$

And τ being 1 at C_0, the minimum disappears only, when τ becomes $= 1$ in the above expression. And this is evidently only the case for $\varphi = \infty$, i. e. when T_1 and T_2 should have the same value. Hence in general there will always be found a minimum in the neighbourhood of C_0. For $\varphi = 1$ we find $\tau = 0{,}970$, $\omega = 0{,}94$, $x = 0{,}506$; for $\varphi = 2$ we find $\tau = 0{,}990$, $\omega = 0{,}98$, $x = 0{,}501$; etc. etc.

It may be easily demonstrated that in the neighbourhood of C_1 such a minimum never appears in our case. For from (1*b*) follows with $\tau = \frac{T_1}{T_0} = \frac{16}{27}\varphi^2$:

$$\frac{16}{27}\varphi^2 = 4\omega\left[x(1-x) + (\varphi + x)^2(1-\omega)^2\right].$$

After substitution of $x = \Delta$, $\omega = {}^1/_3(1+\delta)$, we get, neglecting Δ^2, which is justified by the result:

$$(1+\delta)\left[\frac{9}{4}\frac{\Delta}{\varphi^2} + \left(1 + \frac{2\Delta}{\varphi}\right)(1 - {}^1/_2\delta)^2\right] = 1,$$

or as $\frac{1}{1+\delta} = 1 - \delta + \delta^2 - \dots$, and so $\frac{1}{1+\delta} - (1 - {}^1/_2\delta)^2 = {}^3/_4\delta^2$:

$$\frac{9}{4}\frac{\Delta}{\varphi^2} + \frac{2\Delta}{\varphi}(1 - {}^1/_2\delta)^2 = {}^3/_4\delta^2,$$

yielding:

$$\Delta = {}^3/_4\delta^2 : \left(\frac{9}{4\varphi^2} + \frac{2}{\varphi}\right) = 3\delta^2 : \left(\frac{9}{\varphi^2} + \frac{8}{\varphi}\right).$$

The spinodal line $T = T_1$ touches, therefore, the axis $x = 0$ for every value of φ, and, at least on the assumptions made by us concerning a and b, a minimum can therefore never appear in the neighbourhood of C_1, in consequence of which the spinodal lines in the immediate neighbourhood of C_1 would enclose this point.

Finally some corresponding values of x and ω are subjoined, which determine the shape of the spinodal line $\tau = 1$ ($T = T_0$). By solution of the quadratic equation

$$4\omega\left[x(1-x) + (1+x)^2\,1 - \omega)^2\right] = 1$$

follows immediately:

$\omega = 1$	0,8	0,7	0,6	0,5	0,4	0,33	0,3	0,2	0,1
$x = 0,5$	0,403	0,292	0,227	0,184	0,164	0,182	0,182	0,306	0,679
	0,743	1,004							

So this line cuts the axis $x = 1$ for $\omega = 0,7$, and henceforth only one solution satisfies. x becomes evidently 1 for $\omega(1-\omega)^2 = {}^1/_{10}$, yielding about $\omega = 0,07$.

From the above derived equation $4\varphi(\varphi+1)(1-\omega)^2 + 1 - \tau(2-\omega) = 0$, which was the condition for two equal values of x, we find $\varphi = 1$, $\tau = 1$:

$$8\omega^2 - 15\omega + 7 = 0,$$

from which, besides $\omega = 1$, $\omega = {}^7/_8$ follows. To this belongs then $x = {}^{11}/_{21} = 0,524$. Between $\omega = 1$ and $\omega = 0,875$ we find only imaginary values for x in the above table.

As to the spinodal line $T = T_1$ ($\tau = 0{,}59$), we calculate $x = 0{,}0019$ for $\omega = 0{,}30$, whereas $x = 0{,}006$ corresponds to $\omega = 0{,}40$.

As to the shape of the spinodal lines for great values of v (vapour branch) i. e. when τ and ω approach to 0, follows immediately from (1b):

$$\tau = 4\omega\left[x(1-x) + (\varphi + x)^2\right] = 4\omega\left[\varphi^2 + (2\varphi + 1)x\right].$$

If we substitute $T/T_0 = T : \frac{{}^1/_2 a^2}{Rb}$ for τ, and $\frac{b}{v}$ for ω, we get:

$$RT = \frac{2a^2}{v}\left[\varphi^2 + (2\varphi + 1)x\right].$$

After substitution of $\varphi = \frac{\sqrt{a_1}}{a}$, this becomes:

$$v = \frac{2}{RT}\left[a_1 + (a_2 - a_1)x\right].$$

From this follows that the vapour branches of the spinodal lines in their v, x-projection will approach more and more to straight lines, which will cut the axes $x = 0$ and $x = 1$ at distances proportional to the quantities a_1 and a_2.

5. Let us now consider the second type, which occurs for $\varphi = 2$.

a. *Description of the case* $\varphi = 2$ (fig. 2 and fig. 2a).

The two plaitpoint curves of fig. 1, viz. C_0C_2 and C_1A have met for φ about 1,43 (see fig. 4), after which two new ones have been formed, now C_1C_2 and C_0A. This case, which is found for comparatively large values of φ, for which the proportion $\frac{T_2}{T_1}$ approaches more and more to unity, is the usual one or the normal one. It is the *principal type III*, as described in one of my two preceding papers [1]).

The region of negative pressures on the spinodal lines extends now all over the v, x-diagram, from $x = 0$ to $x = 1$, and is bounded by the two dotted curves (see fig. 2) above and below.

The spinodal line belonging to $\tau = 1{,}35$ touches now the curve C_0A in the point R_2. Again the plaitpoints are not realisable from a point between R_2 and C_0 to A (see the footnote in § 4 at **a.**)

Beyond R_2 the temperature and with it the pressure decreases, so that in the p,T diagram (see fig. 2a) the curve C_0R_2A *runs back*

[1]) l. c. p. 642—644.

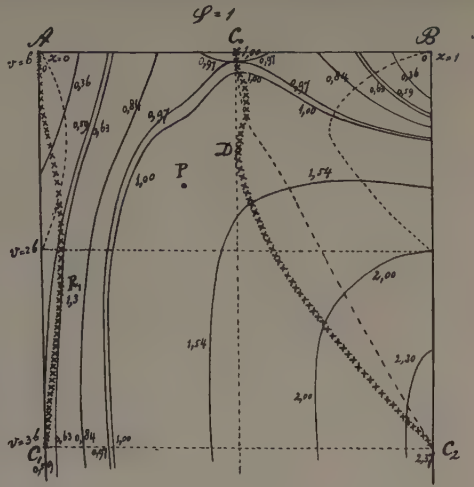

Fig. 1.

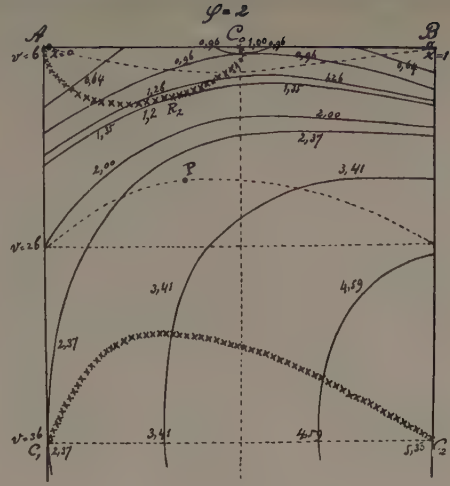

Fig. 2.

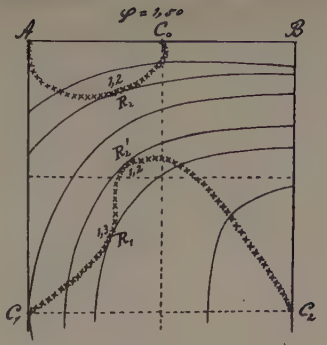

Fig 3.

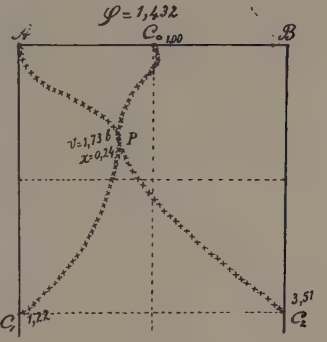

Fig 4.

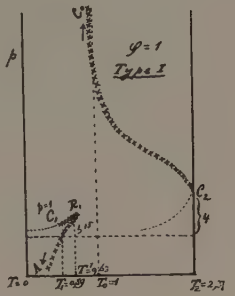

Fig 1a.

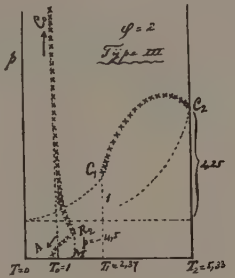

Fig. 2a.

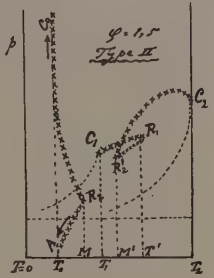

Fig. 3a.

again from R_2. In R_2 the pressure is already negative, and it becomes again $= -27\,p_1$ in A. (See § 4 at **b**).

When $\varphi = 2$, we find easily from the equations derived in § 3, that then $T_2/T_1 = 2^1/_4$ and $T_1/T_0 = {}^{64}/_{27}$. So if T_0 is again $= 1$, then $T_1 = 2{,}37$ and $T_2 = 5{,}33$. Now T_1 is higher than T_0.

b. *Some mathematical and numerical details.*

Much having already been derived in § 4, it will suffice to give some few values.

Of the two plaitpoint curves the following points were calculated

$\varphi = 2$

$x = 0$	0,15	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	
$\omega = 0{,}33^8$	$0{,}39^5$	$0{,}40^3$	$0{,}41+$	$0{,}41-$	0,40	$0{,}39+$	$0{,}37^4$	$0{,}35^9$	$0{,}34^7$	$0{,}33^8$	Curve C_1C_2
$\tau = 2{,}37$	2,87	3,04	3,38	3,70	4,00	4,28	4,60	4,87	5,10	5,33	
$\pi = 1$	1,46	1,62	1,90	2,11	2,25	2,32	2,37	2,36	2,33	2,25	

$x =$	0	0,01	0,1	0,2	0,3	0,4	0,5	
$\omega =$	1	0,91	0,81	0,78	0,80	0,85	0,933and1	Curve C_0A.
$\tau =$	0	$0{,}15^5$	0,81	1,23	1,35	1,26	1,04 and 1	
$\pi =$	−27	−17,3	−7,90	−5,16	−4,62	−3,98	49 and ∞	

The separation between the negative and positive pressures on the spinodal curves is given by

$\omega =$	1	0,9	0,894	0,606	0,6	0,5
$x =$	0	0,31	0,40	0,40	0,31	0
	1	0,50	0,40	0,40	0,50	1

The places where x has here two equal values, are easily found from the value of x given in § 4. Evidently we must have then $\theta = (1-\omega)(2\omega - 1) = {}^1/_{12}$. This gives $\omega = 0{,}894$ and $0{,}606, = {}^1/_4(3 \pm {}^1/_3\sqrt{3})$. For $\varphi = 1$ θ would have to be $^1/_4$, and there are no values of ω which satisfy this condition.

For the calculation of the different spinodal curves it is convenient to know the limiting values of τ again. We find for $x = 0$:

$v/b = {}^1/_\omega =$	1	1,25	1,50	1,75	2	2,25	2,50	2,75	3
$\tau =$	0	0,51	1,19	1,68	2	2,20	2,30	2,36	2,37

For $x = 1$ these values are all $2^1/_4$ times greater.

For $x = {}^1/_2$ we find with the same values of ω:

$\tau =$ 1	1,60	2,53	3,20	3,63	3,88	3,99	4,04	4,04

$\omega = 1$ yields the same values as in § 4 for $\varphi = 1$.

$\omega = {}^1/_2$ yields:

$x = 0$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\tau =$ 2,37	2,73	3,08	3,41	3,73	4,03	4,33	4,59	4,86	5,10	5,33

6. We may now determine, where the transition represented in fig. 4, takes place. (The place of the point P is also drawn in figs. 1 and 2 [1]).

If we put $1 - \omega = y$ in the equation (2b) of the plaitpoint curve, then

$$(1-2x) + (x+\varphi)\,y^2\left(3 + (x+\varphi)^2\,\frac{y\,(3y-2)}{x\,(1-x)}\right) = 0. \quad . \quad . \quad . \quad (a)$$

Now in the double point sought $\frac{\partial f}{\partial x}$ must be 0 and $\frac{\partial f}{\partial y}$ must be 0, when f denotes the first member of (a). This gives:

$$-2x\,(1-x) + (1-2x)^2 + 3y^2\left\{(1-2x)\,(x+\varphi) + x\,(1-x)\right\} +$$

$$+ 3\,(x+\varphi)^2\,y^3\,(3y-2) = 0\ , \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and after division by $6y\,(x+\varphi)$:

$$x\,(1-x) + (x+\varphi)^2\,y\,(2y-1) = 0 \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Substitution of the value of $x\,(1-x)$ from (c) in (a) gives:

$$(1-2x) + (x+\varphi)\,y^2\left(3 + \frac{3y-2}{1-2y}\right) = 0,$$

or

$$(1-2x) + (x+\varphi)\,y^2\,\frac{1-3y}{1-2y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (a')$$

So we have to solve y, x and φ from (a'), (b) and (c). Substitution of $1-2x$ from (a'), and $x\,(1-x)$ from (c) in (b) gives, after division by $(x+\varphi)^2 y$:

$$-2\,(1-2y) + y^3\left(\frac{1-3y}{1-2y}\right)^2 +$$

$$+ 3y^2\left\{-y\,\frac{1-3y}{1-2y} + (1-2y)\right\} + 3y^2\,(3y-2) = 0,$$

i. e. after multiplication by $(1-2y)^2$:

$$-2\,(1-2y)^3 + y^3\,(1-3y)^2 +$$

$$+ 3y^2\,(1-2y)\left\{-y\,(1-3y) + (1-2y)^2\right\} + 3y^2\,(3y-2)\,(1-2y)^2 = 0,$$

from which y may be solved. The above equation gives:

[1]) This point must be thought more to the left. In fig. 4 no contact but intersection takes place in the double point P.

$$-2(1-2y)^3 + y^3(1-3y)^2 + 3y^2(1-2y)(y^2+2y-1) = 0,$$

or $$3y^5 - 15y^4 + 29y^3 - 27y^2 + 12y - 2 = 0,$$

i. e. after division by $(y-1)^3$:

$$3y^2 - 6y + 2,$$

yielding:

$$y = 1 \pm {}^1/_3\sqrt{3}.$$

As it is obvious that y cannot be larger than 1, only:

$$\underline{y = 1 - {}^1/_3\sqrt{3} = 0{,}4226}$$

satisfies here.

If we substitute the value $x+\varphi$ from (a') into (c), we get:

$$x(1-x) - (1-2x)^2\frac{(1-2y)^3}{y^3(1-3y)^2} = 0.$$

In this the last fraction passes into ${}^1/_4(1+\sqrt{3})$, after substitution of $y = 1 - {}^1/_3\sqrt{3}$, so that we get for x:

$$x(1-x) - {}^1/_4(1+\sqrt{3})\left\{1 - 4x(1-x)\right\} = 0,$$

hence:

$$x(1-x) = {}^1/_4(-1+\sqrt{3}),$$

giving:

$$x = {}^1/_2\left\{1 \pm {}^1/_2(\sqrt{6}-\sqrt{2})\right\} = 0{,}2412 \text{ or } 0{,}7588.$$

It is obvious from the figure, that only the first value satisfies, viz.:

$$\underline{x = {}^1/_2\left\{1 - {}^1/_2(\sqrt{6}-\sqrt{2})\right\} = 0{,}2412.}$$

The value of φ is finally found from (c):

$$(x+\varphi)^2 = \frac{x(1-x)}{y(1-2y)} = {}^3/_4(2+\sqrt{3}),$$

giving $x+\varphi = {}^1/_4(3\sqrt{2}+\sqrt{6})$, hence $\underline{\varphi = {}^1/_2(-1+\sqrt{2}+\sqrt{6}) = 1{,}432.}$

As $y = 1 - {}^1/_3\sqrt{3}$, $\omega = {}^1/_3\sqrt{3}$, i. e. the intersection takes place at $v = b\sqrt{3} = 1{,}732\,b$.

As mentioned before $T_0 = T_1$ for $\varphi = 1{,}30$ (see § 3). For $\varphi = 1{,}43$ T_0 is already $< T_1$. For ${}^{T_1}/_{T_0} = {}^{16}/_{27}\varphi^2$ we find easily the value 1,215, while 2,887 is found for ${}^{T_2}/_{T_1} = (1+1/\varphi)^2$.

7. Besides the cases, given in figs. 1 and 2, representing the principal types I and III, there is another important type, viz. II, of which I also gave a full description in my previous paper, which I have already cited several times [1]). The p,T-diagram of this case

[1]) l. c. p. 663—667.

is given in fig. 4*a*. KUENEN met with it, among others, in the case of mixtures of C_2H_6 with ethyl- and some higher alcohols. Also triethylamine with water is a well-known instance.

This case is evidently found, when the plaitpoint curve C_1C_2 of fig. 2 assumes the shape drawn in fig. 3. We may namely imagine that when the two curves C_1C_2 and C_0A approach each other, a deviation from the straight course may be found on the left side of C_1C_2, specially if b_1 should not be $= b_2$, by which the point C_0 would therefore be shifted to the left, to the side of the small volumes. At all events the anomaly of one of the two components can give rise to the occurrence of this second principal type, as I showed in a preceding paper.

From the shape of the different spinodal curves it is obvious that from C_1 the temperatures first increase, as far as the point of contact at R_1. The temperature is then T' (see fig. 3*a*). But between R_1 and R_2', where the plaitpoint curve is again touched by one of the spinodal curves, the temperature *decreases*, and so also the pressure, so that in the p, T-diagram of fig. 3*a* the line R_1R_2' *runs back* again, as in fig. 1*a* the line R_1A and in fig. 2*a* the line R_2A, having in this case two *cusps* in R_1 and R_2'.

Here the points between R_1 and R_2', and also those on C_1R_1 and C_2R_2' in the neighbourhood of R_1 and R_2' can again not be realized, and the consequence will be the occurrence of a three phase equilibrium[1]).

As I already observed in one of my previous papers (l.c. p. 646), after the two liquid phases 1 and 2 have coincided in the neighbourhood of the point R'_2, here too, separation of the two liquid phases must take place again — provided the temperature be sufficiently lowered — and this will take place in the neighbourhood of the point, where one of the spinodal curves in R_2 touches the plaitpoint curve C_0A. This is also represented in the p, T-diagram of fig. 3*a*.

When comparing figs. 1, 2 and 3, we see clearly the connection between the three principal types and their transition into each other. The connection is given by the different course of the two plaitpoint curves in figs. 1 and 2, which (see fig. 3) may pass *continuously* into each other with changed circumstances of critical data of the two components.

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[1]) Cf. VAN DER WAALS, Continuität II, p. 187, and These Proceedings V, p. 307—11 Oct. 25, 1902.

Physics. — "*Some remarks on* Dr. PH. KOHNSTAMM's *last papers.*" By J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

1. With interest and full approval I read Dr. KOHNSTAMM's three papers on the osmotic pressure [1]). From them it appeared to me that, practically, he perfectly agreed with me. Only with regard to a few points there are differences of opinion — only *in appearance*, however, as I shall show in what follows.

On pages 723—729 l. c., namely, KOHNSTAMM gives also a *thermodynamic* derivation of the osmotic pressure, which seems to lead to a somewhat different result from mine. He finds, namely, in the numerator finally the quantity $v_x - x\frac{db}{dx}$ instead of v_0. [I use here my notation; v_0 is the molecular volume of the pure solvent (KOHNSTAMM's v_g), v_x that of the solution, in which the dissolved substance is present with a concentration x (K.'s v_0)]. But here he overlooks that according to his approximations v_0 may be written for the latter. For on page 726 an integral is neglected, among others on the strength of the fact that $v_x - b$ approaches to 0. He puts therefore $v_x = b$, in consequence of which $v_x - x\frac{db}{dx} = b - x\frac{db}{dx} = = b - x(b_2 - b_1) = b_1$. This however, is the value of b or v, when $x = 0$, so v_0.

So KOHNSTAMM finds exactly the same thing as I found already in 1894 in a much simpler way. In my method no integral need be split into three parts, and we need not neglect anything but the compressibility of the liquid (which is of course also done by KOHNSTAMM), so that my result (the compressibility excepted) is *perfectly accurate,* which cannot be said of that of KOHNSTAMM.

2. The above mentioned method has been repeatedly published by me. [Z. f. Ph. Ch. XV, 1894; Arch. Teyler (Théorie générale), 1898; Lehrbuch der math. Chemie, 1901; Arch. Teyler (Quelques remarques sur la théorie des solutions non-diluées), 1903; and recently in the "Chemisch Weekblad", 1905, N°. 9]. The derivation may follow here once more.

If there is namely, equilibrium between the solution with the concentration x under a pressure p, with the pure solvent with a concentration 0 under the *arbitrary* pressure p_0 (e.g. that of the saturated vapour, or of the atmosphere etc.), the molecular potentials

[1]) These Proc. VII, 723—751.

of the solvent in the two liquid phases (separated by a semipermeable membrane only passable by the solvent) are the same. Hence:

$$\mu(0, p_0) = \mu(x, p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But evidently we have the identity

$$\mu(0, p_0) = \mu(0, p) - \int_{p_0}^{p} \frac{\partial \mu_0}{\partial p}\, dp\,.$$

Here $\frac{\partial \mu_0}{\partial p} = v_0$ (for meaning of v_0, see §1). So we have also:

$$\mu(0, p_0) = \mu(0, p) - \int_{p_0}^{p} v_0\, dp\,.$$

If we now assume v_0 to be *independent of the pressure* — which KOHNSTAMM thinks perfectly permissible — we get:

$$\mu(0, p_0) = \mu(0, p) - v_0 (p - p_0)\,.$$

Substituting this in (1), we get at once:

$$\pi = p - p_0 = \frac{1}{v_0} (\mu_0 - \mu_x)_p\,, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

by which the osmotic pressure is immediately brought into connection with the difference of the molecular potentials of the pure solvent and of that in the solution, *both under the same pressure p.*

Now we can in the usual way replace $\mu_0 - \mu_x$ by its value. We find then, as has been frequently derived:

$$- RT \log (1 - x) - \frac{\alpha x^2}{(1 + rx)^2} + RT \log \frac{v_x - b}{v_0 - b_1} \text{ etc.}$$

in which the latter terms is often neglected, and α and r have the known meaning.

In this way the apparent deviation with regard to v_0 has been disproved. My statement, therefore, that in the numerator for v_0 no correction term need be applied (see KOHNSTAMM, p. 729), was by no means "too absolute".

3. When reading through KOHNSTAMM's paper, I was further struck by the following in my opinion inaccurate assertions.

On p. 739 it says: "It appears from the explanation convincingly, that VAN LAAR goes too far, when he states, that we cannot speak of osmotic pressure in an *isolated* solution."

I fully maintain this view. For in the kinetic explanation of

KOHNSTAMM the osmotic pressure in an isolated solution is established, *only when* he places semi-permeable walls or planes in it. But then it is of course no isolated solution any more! What I demonstrate is no more than this: Without semipermeable membrane no osmotic pressure. And to this KOHNSTAMM will certainly not have any objection, witness the cited question of PUPIN how it is possible, that e. g. a $CaCl_2$-solution of no less than 53 atm. could be held in a thin glass vessel without bursting it! I do not see very well, what objection KOHNSTAMM can have to my assertion. For this is the *core* of the question, with regard to which he proves to be quite of my opinion in another place (cf. p. 742).

4. What KOHNSTAMM further observes on pages 742—4 with regard to the idea "thermodynamic potential", and what he says on "palpable conceptions" may be very well left undiscussed here. For this is only a question of words, which does not affect the real nature of the affair at all. Every one who works with the thermodynamic potential, means with it the ζ-function of GIBBS, which perfectly determines the condition of equilibrium, as it must be *minimum* in this case.

Finally I may only be allowed to point out that Dr. KOHNSTAMM has evidently misunderstood me, where he says that he thinks the request to supply something "as a substitute" for the osmotic pressure and the kinetic conception of it less unreasonable than it seems to *me* (p. 746).

I, namely, spoke of the osmotic pressure in an *isolated* solution. And I very distinctly added: nothing can be put in the place for what *does not exist.* And I wrote further, that the *usual* (faulty) *kinetic* conception of the osmotic pressure (i. e. where there are semipermeable membranes) must be replaced by a perfectly new kinetic explanation, in which inter alia, the process of diffusion at the membrane is put more into the foreground (Ch. Weekbl., 1905, N°. 9).

And where KOHNSTAMM himself has made a very laudable attempt in this direction (l. c. p. 729—741) to explain the osmotic pressure, I have after all reasons for satisfaction, though he has wisely abandoned the idea of drawing up an equation for non-diluted solutions in *this* way.

And as to the *thermodynamic* derivation, in this KOHNSTAMM has been less fortunate in my opinion; where he has tried to substitute for my perfectly exact, and yet so simple derivation an indirect, elaborate derivation, the result of which on account of some neglections cannot even lay claim to perfect accuracy.

Mathematics. — "*On the rank of the section of two algebraic surfaces.*" By Dr. W. A. VERSLUYS. (Communicated by Prof. P. H. SCHOUTE.

1. In this paper I intend to prove the relation new to me

$$r = m_1 n_2 + m_2 n_1 - 2\delta - 3\chi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where r is the rank of the curve of intersection s of two algebraic surfaces S_1 and S_2, respectively of the degree n_1 and n_2 and of the class m_1 and m_2 and possessing in δ points an ordinary contact and in χ points a stationary contact. Some applications of this formula are given too.

Formerly I proved [1]) the following extension of a well-known formula [2])

$$r = n_1 n_2 (n_1 + n_2 - 2) - 2(n_1\xi_2 + n_2\xi_1 + \delta) - 3(n_1\nu_2 + n_2\nu_1 + \chi), \quad . \quad (B)$$

where $\xi_1, \xi_2, \nu_1, \nu_2$ represent the degrees of the nodal and cuspidal curves of the two surfaces S_1 and S_2. Formula (A) shall first be proved for the case that S_1 and S_2 are developables. If we wish to apply formula (B) to developables the numbers of double generating lines ω_1 and ω_2 must be added to the orders ξ_1 and ξ_2 of the nodal curves and the numbers of stationary generating lines v_1 and v_2 to the orders ν_1 and ν_2 of the cuspidal curves.

Formula (B) becomes

$$r = n_1 n_2 (n_1 + n_2 - 2) - 2\{n_1(\xi_2 + \omega_2) + n_2(\xi_1 + \omega_1) + \delta\} - \\ - 3\{n_1(\nu_2 + v_2) + n_2(\nu_1 + v_1) + \chi\} \quad . \quad . \quad . \quad . \quad (C)$$

2. Let $\Delta^2 S$ be the second polar surface of the degenerated surface $S_1 + S_2$ with respect to the arbitrary point P. This surface $\Delta^2 S$ is of the degree $(n_1 + n_2 - 2)$ and meets the curve of intersection s of S_1 and S_2, this curve being of the degree $n_1 n_2$, in $n_1 n_2 (n_1 + n_2 - 2)$ points.

These points of intersection are 1st the triple points of $S_1 + S_2$ through which the curve s passes and 2nd the points of s for which the tangent plane to one of the two surfaces passes through P. The triple points of $S_1 + S_2$ through which the curve s passes are the points in which a double line of one of the two surfaces meets the other surface. So these triple points are:

1st. The $(n_1\nu_2 + n_2\nu_1)$ points in which a cuspidal curve of one of the surfaces meets the other one. These points are cusps on the curve of intersection s, they are indicated by CREMONA as points λ

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[1]) VERSLUYS, Mémoires de Liége, 3me serie, t. VI. Sur les nombres Plückériens etc.

[2]) E. PASCAL, Rep. di Mat. Sup. II, p. 325.

and must count according to him for three points of intersection of the nodal curve, thus here of the curve s with $\Delta^2 S$ [1]).

2nd. The $(n_1 v_2 + n_2 v_1)$ points in which a stationary generating line of one of the surfaces meets the other one. These points, also cusps on the curve s, are indicated by CREMONA as points r which must count according to him for three points of intersection of the nodal curve s with $\Delta^2 S$ [2]).

3rd. The $(n_1 \xi_2 + n_2 \xi_1)$ intersections of S_1 or S_2 with the nodal curve of the other surface. According to CREMONA each of the branches of the nodal curve meets $\Delta^2 S$ [3]) one time in a triple point τ. Through each of these points τ pass two branches of s, which is a nodal curve on $S_1 + S_2$; so each of these triple points counts for two points of intersection of s with $\Delta^2 S$.

4th. The $(n_1 \omega_2 + n_2 \omega_1)$ nodes of s in which a double generator of one of the surfaces S_1 or S_2 meets the other one. According to CREMONA the nodal curve s meets $\Delta^2 S$ [4]) two times in such a triple point r.

The surface $S_1 + S_2$ possesses still more triple points, among others the cusps β of the cuspidal curves; these points do lie on $\Delta^2 S$, but on the curve of intersection s they do not; so they do not belong to the points of intersection of s with $\Delta^2 S$.

3. Through P pass m_1 tangent planes of the surface S_1. A generator of S_1, along which one of the m_1 tangent planes through P touches S_1, meets n_2 times the surface S_2. Each of these points of intersection is a point on s also situated on $\Delta^2 S$. Such a point of s and of $\Delta^2 S$ counts for *one* point of intersection, according to CREMONA [5]). So $\Delta^2 S$ is met by the curve s in $(m_1 n_2 + m_2 n_1)$ points for which one of the tangent planes passes through P.

This gives the relation:

$$n_1 n_2 (n_1 + n_2 - 2) = m_1 n_2 + m_2 n_1 + 2\{n_1 (\xi_2 + \omega_2) + n_2 (\xi_1 + \omega_1)\} + \\ + 3\{n_1 (v_2 + v_2) + n_2 (v_1 + v_1)\} \quad . \quad . \quad . \quad . \quad . \quad (D)$$

Comparing the equations (C) and (D) we get immediately

$$r = m_1 n_2 + m_2 n_1 - 2\delta - 3\chi. \quad . \quad . \quad . \quad . \quad . \quad (A)$$

The degree of a developable being the rank of its cuspidal curve, we can write for this formula:

$$r = m_1 r_2 + m_2 r_1 - 2\delta - 3\chi.$$

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1) CREMONA—CURTZE, Oberflächen § 108.
2) CREMONA—CURTZE, loc. cit. § 100.
3) CREMONA—CURTZE, loc. cit. § 109.
4) CREMONA—CURTZE, loc. cit. § 101.
5) CREMONA—CURTZE, loc. cit. § 99.

4. The formula (D) and hence also the formula (A), which is now proved for the case that the two surfaces are developables holds still good when S_1 and S_2 are arbitrary algebraic surfaces. Let ξ_1 and ν_1 represent the degree of the total nodal curve and total cuspidal curve of S_1, likewise ξ_2 and ν_2 for S_2. One of the formulae of PLÜCKER applied to an arbitrary plane section of S_1 gives

$$m_1 = n_1^2 - n_1 - 2\,\xi_1 - 3\,\nu_1,$$

or

$$0 = n_1^2 - n_1 - m_1 - 2\,\xi_1 - 3\,\nu_1.$$

In like manner an arbitrary plane section of S_2 gives

$$0 = n_2^2 - n_2 - m_2 - 2\,\xi_2 - 3\,\nu_2$$

hence

$$0 = n_2\,(n_1^2 - n_1 - m_1 - 2\,\xi_1 - 3\,\nu_1) + n_1\,(n_2^2 - n_2 - m_2 - 2\,\xi_2 - 3\,\nu_2)$$

or

$$n_1 n_2 (n_1 + n_2 - 2) = m_1 n_2 + m_2 n_1 + 2(n_1\xi_2 + n_2\xi_1) + 3(n_1\nu_2 + n_2\nu_1). \ldots (D')$$

combining the formulae (D') and (B) we get the formula (A).

If S_2 is a plane, n_2 becomes equal to unity and m_2 equal to nought, whilst the curve s becomes a plane section and the rank r of s passes into the class of the plane section. So formula (A) gives for that class

$$r = m_1 - 2\,\delta - 3\,\chi,$$

which is indeed the class of a section of S_1 with a plane, having with S_1 in δ points an ordinary contact and in χ points a stationary contact.

5. If S_2 is of the second degree and S_1 of the degree n and of the class m, the formula (A) gives for the rank of the curve of intersection

$$r = 2\,(m + n) - 2\,\delta - 3\,\chi.$$

If S_2 is a quadratic cone K^2 this formula will be proved directly once more as follows for the sake of verification.

The rank of the curve of intersection s is the number of its tangents meeting an arbitrary right line, e.g. a generator l of K^2. Each tangent of s, meeting the generator l has three points in common with the cone K^2, in fact the two consecutive points it has in common with s and its point of intersection with l, unless the latter coincides with the point of contact to s. Each right line having three points in common with K^2 lies entirely on K^2. The only tangents of s meeting l are thus the generating lines of K^2 which are at the same time tangents of s and the tangents to s at its points of intersection with l. The generator l of K^2 meets S_1 and therefore s too n times; through each of these points of intersection

pass two consecutive tangents of s. Whence already $2n$ tangents of s meeting l.

Tangents of s, being at the same time generating lines of K^2, pass through the vertex T of K^2 and, being tangents of s, are also tangents of S_1, and therefore situated on the tangent cone K of S_1, having T for its vertex. Conversely every common generator of the two cones K^2 and K is a generator of K^2 having with S_1, thus also with s, two coinciding points in common. A right line having with s two coinciding points in common is either a tangent of s or it passes through a double point of s. So the common generators of the cones K^2 and K are either tangents of s or they pass through double points of s. The order of the tangent cone K, being equal to the class m of S_1, the number of common generators is $2m$. The number of tangents of s meeting l in the vertex T will be $2m$, diminished by a number still to be determined for the common generators passing through a double point of s.

If K^2 has in a point δ an ordinary contact with S_1 the common tangent plane π in δ is a tangent plane of S_1 passing through T. So π is also a tangent plane to the cone K along the line $T\delta$. So the two cones K^2 and K have along the common generator $T\delta$ a common tangent plane. The line $T\delta$ must therefore count for two common generators of the cones K^2 and K. A point δ is a node of s and with the exception of very particular cases the two tangents of s in δ will not coincide with $T\delta$. So for every point δ the number of tangents of s passing through T must be diminished by two.

The following example proves that for every point χ in which S_1 and K^2 have a stationary contact, the number of generators of K^2 touching s must be diminished by three. Let S_1 also be a quadratic surface and let the curve of intersection s be a not degenerated biquadratic curve R^4 with a cusp χ. Then the line $T\chi$ counts already at least for two common generators of the cones K^2 and K and is again not a tangent in χ to s or R^4. If now $T\chi$ were to count only for two common generators the cones K^2 and K would have two more generators in common. These latter two cannot be two consecutive generators, for in that case R^4 would have two double points and so it would have to break up. Now it is easy to see that these two remaining generators are tangents to R^4 or s at points for which the osculating plane is a stationary plane. So R^4 would have to possess two stationary planes α whilst a R^4 with cusp possesses but one stationary plane α [1]). The right line $T\chi$ must

[1]) E. Pascal, loc. cit. p. 363.

therefore count for three common generators of K^2 and K. The number of tangents of s meeting the line l, thus the rank of s is consequently

$$r = 2n + 2m - 2\delta - 3\chi .$$

6. The reciprocal polar figure s' of the curve of intersection s of K^2 and S_1 is a developable circumscribed to a conic c^2 and to a surface S' of order m and of class n, whilst the conic c^2 touches δ times the surface S' and osculates it χ times. If we take for the conic c^2 the imaginary circle at infinity the developable s' becomes the developable focal surface of S'. The rank of s' is the same as that of s. So we find the theorem:

The rank of the focal developable of a surface of order m and of class n touching the imaginary circle at infinity δ times and osculating it χ times is

$$r = 2m + 2n - 2\delta - 3\chi.$$

If S_1 is a developable the point of contact of a common tangent plane that is an ordinary plane of S_1 is always a node of s [1]). The developables K^2 and S_1 will only then have a stationary contact in a point χ, when the common tangent plane is a stationary plane α of S_1. The line $T\chi$ counts thus for four lines of intersection of the cone K^2 with the tangent cone K which breaks up into m planes. It is easy to see that now the line $T\chi$ is at the same time tangent to s at the special cusp χ which is a singularity of order two, of rank unity and of class three [2]). So a stationary contact χ gives rise to four lines of intersection of K^2 with K of which only one is an ordinary tangent of s lying on K^2. Each point χ now also diminishes the rank of s by three. The reciprocal polar figure of S_1 is a curve S' of order m and of class n. Each common tangent plane of K^2 and S_1 is transformed in a common point of c^2 and S'. If the common plane is a stationary plane α of S_1 the common point is a cusp on the curve S'. So we find the theorem:

The rank of the focal developable of a plane curve or a twisted curve of the degree m and of the class n and of which δ ordinary points and χ cusps lie on the imaginary circle at infinity is

$$r = 2m + 2n - 2\delta - 3\chi.$$

7. If S'_1 and S'_2 are the reciprocal polar figures of the surfaces S_1 and S_2, then S'_1 and S'_2 are respectively of the degree m_1 and m_2 and of the class n_1 and n_2.

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[1]) VERSLUYS, Mémoires de Liège, 3me série t. VI. loc. cit.

[2]) HALPHEN, Bull. de la Soc. Mat. de France, t. VI, p. 10.

If the surfaces S_1 and S_2 have an ordinary contact in δ points, the common tangent planes in these δ points are ordinary double tangent planes of the developable D circumscribing S_1 and S_2 [1]). The surfaces S'_1 and S'_2 will also have in δ points an ordinary contact.

If the surfaces S_1 and S_2 have in χ points a stationary contact the tangent planes in these χ points are stationary tangent planes of the developable D [2]). The surfaces S'_1 and S'_2 have thus also in χ points a stationary contact.

So the rank of the curve of intersection d' of the surfaces S'_1 and S'_2 is according to formula (A), just as the rank of the curve s,

$$r = m_1 n_2 + m_2 n_1 - 2\delta - 3\chi.$$

The curve d' being the reciprocal polar figure of the circumscribing developable D, the rank of D is equal to the rank of d'. Whence the theorem:

For two arbitrary algebraic surfaces the rank of the curve of intersection is equal to the rank of the circumscribing developable.

Here we have supposed that the points of contact δ and χ are ordinary points on both surfaces and the tangent planes ordinary tangent planes in these points [3]).

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[1]) VERSLUYS, Mém. de Liège. 3me série t. VI. De l'influence d'un contact etc.

[2]) VERSLUYS, loc. cit.

[3]) VERSLUYS, loc. cit.

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